Real intersection theory I

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Abstract

Let $\mathcal{X}$ be a C^∞ manifold equipped with a covering called de Rham data. Let $\mathscr{D}'(\mathcal{X})$ be the linear space of currents. Measure-theoretically, we construct a subspace $\mathscr{L}(\mathcal{X}) \subset \mathscr{D}'(\mathcal{X})$ and a bilinear map, called current's intersection,

$$\begin{array}{ccc} \mathscr{L}(\mathcal{X}) \times \mathscr{L}(\mathcal{X}) & \rightarrow & \mathscr{L}(\mathcal{X}) \\ (T_1, T_2) & \rightarrow & [T_1 \wedge T_2] \end{array}.$$

The intersection is dependent of de Rham data. However it has a rich structure that form the real intersection theory. In the part I (this paper), we prove the existence of such an intersection.

Contents

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1 Introduction

1.1 History of the current's intersection

Inspired by the original formulation of de Rham theory, we interpret a particular type of weak limits of measures as an intersection in geometry.

It begins with de Rham's work in differential geometry. Let $\mathcal{X}$ be a C^∞ manifold of dimension m. In [3], G. de Rham defined the intersection current

$$T \wedge \omega \tag{1.1}$$

between a current T and a C^∞ form ω, expressed as a functional on $\mathscr{D}(\mathcal{X})$ – the space of C^∞ differential forms with a compact support,

$$\int_T \omega \wedge (\bullet) \tag{1.2}$$

where the integral notation $\int_T (\bullet)$ denotes the functional. The intersection satisfies

$$supp(T \wedge \omega) \subset supp(T) \cap supp(\omega) \tag{1.3}$$

where $supp(\cdot)$ stands for the support. The asymmetric expression (1.1) led to the symmetric completion that historically emerged into the topology. For instance, G. de Rham extended (1.1) to the intersection number between two currents,

$$T \wedge S[1], \tag{1.4}$$

where S is another current of dimension $m - dim(T)$. To do that, he first constructed the de Rham's regularization $R_\epsilon T$ that is a family of C^∞ forms for real $\epsilon > 0$, converging to T as $\epsilon \to 0$. Then he studied the convergence of the real numbers,

$$\int_{\mathcal{X}} R_\epsilon(T) \wedge R_{\epsilon'}(S), \quad as\ (\epsilon, \epsilon') \to (0, 0). \tag{1.5}$$

Such a formulation encountered two obstacles: 1) the sequence is dependent of the non-canonical regularization, 2) the limit may diverge due to the singular support. He overcame them by creating a homotopy to evade. The result is topological, thus weaker than the geometric setup. But it led to the cap product in homology, which later was replaced by the cup product in cohomology. As the cohomological approach prevails, the de Rham's regularization fades out.

1.2 New direction

We return to the de Rham's regularization, but in the new tool of measure theory. In our formulation, we consider the convergence of similar real numbers,

$$\int_{T_1} R_\epsilon T_2 \wedge \phi, \qquad as\ \epsilon \to 0 \tag{1.6}$$

for a fixed test form $\phi \in \mathscr{D}(\mathcal{X})$, where T_1, T_2 are currents satisfying

$$dim(T_1) + dim(T_2) \geq m.$$

The same obstacles still exist. But we do not use a homotopy to evade the divergence. Instead we consider the reason of divergence. We found the divergence lies in the measure of the singular support. So we go straight into the singular support to obtain the convergence in Lebesgue measure. We call this type of currents Lebesgue currents. If one regards geometric measure theory as a method to measure the sets with tangential directions, real intersection theory is a method to intersect such sets. The method has two steps: 1) convert the currents to Lebesgue measure; 2) intersect the measure by taking the limit. Thus the convergence is the weak convergence in Lebesgue measure. For instance our intersection exists only in Lebesgue measure. But the application lies in its connection with the classical cases which already include wedge product of forms, transversal intersection of singular cycles, the proper intersection of algebraic cycles and more. In this paper, we would like to prove a sufficient condition for the convergence. Applying it we obtain a bilinear homomorphism denoted by $[\cdot \wedge \cdot]$,

$$\begin{array}{ccc} \mathscr{L}(X) \times \mathscr{L}(X) & \rightarrow & C(X) \\ (T_1, T_2) & \rightarrow & [T_1 \wedge T_2], \end{array} \tag{1.7}$$

for the subspace $\mathscr{L}(X)$-the collection of Lebesgue currents. So (1.7) does not only extend the formula (1.1), but also (1.3)

$$supp([T_1 \wedge T_2]) \subset supp(T_1) \cap supp(T_2). \tag{1.8}$$

The motivation (discussed elsewhere) is based on our belief that the singular support contains the more advanced structure which is lost in the de Rham's homotopy.

We organized the rest as follows. In section 2, we introduce and explore a particular type of currents called Lebesgue currents. In section 3, we review de Rham's regularization and give a further description of its kernel. In section 4, we show that the conditions for Lebesgue currents are sufficient for the convergence of de Rham's regularization. This leads to the definition of the intersection of currents – so called the intersection of currents.

2 Lebesgue currents

Definition 2.1. *(of notations)*

(1) If T is current and ϕ is a test form, the functional for currents also denoted by $T(\phi)$. The integral notion as in (1.2) will also be used with the focus on the computation.

(2) *The functional of a distribution* $\mathcal{F}$ *is denoted by*

$$\int_{\mathcal{F}} (\bullet) d\mu \tag{2.1}$$

where $d\mu$ *is the Lebesgue measure of the Euclidean space. The notation is extended to the functional of a signed measure that can evaluate characteristic functions of measurable sets, or simply the measurable sets.*

(3) *Let* $\mathbb{R}^m$ *be equipped with the coordinates* $x = \{x_1, \cdots, x_m\}$ *referred to as a chart. Let* V_I *be an* r *dimensional coordinates plane with multi -index* I *of length* r,

$$\pi_I : \mathbb{R}^m \to V_I$$

be the projection. Let $V_{I^\diamond}$ *be the perpendicular coordinates plane of dimension* $m-r$ *satisfying* $\{II^\diamond\} = \{1, 2, \cdots, m\}$ *with concordant orientations. Let* $d\mu^I, d\mu^{I^\diamond}$ *be their Euclidean volume forms*

$$dx_{i_1} \wedge \cdots \wedge dx_{i_r}, dx_{i_1^\diamond} \wedge \cdots \wedge dx_{i_{m-r}^\diamond}$$

with the matching orders of the $\wedge$ *products. Throughout this paper Euclidean volume forms associated with the chart are used in two different ways interchangeably: a) as a* C^∞ *differential form with concordant wedge product, b) as the Lebesgue measure with respect to the chart. For instance* V_I *is equipped with the Lebesgue measure* $d\mu^I$.

(4) *Let* T *be a current of dimension* r *with a compact support in* $\mathbb{R}^m$. *In [3] (Chapter III, §8, p36)* T *is written as*

$$T = \sum_I \mathcal{T}_I d\mu^{I^\diamond} \tag{2.2}$$

the form with distribution values. We call $\mathcal{T}_I$ *for each* I *the de Rham distribution of* T.

(5) *Continuing from (3), let* $\mathcal{T}_{I_1}$ *be one of de Rham distributions among finitely many* $\mathcal{T}_I$. *Then* $(\pi_I)_*(\mathcal{T}_{I_1} d\mu^{I^\diamond})$ *is a current of maximal degree in the plane* V_I *(where* I_1, I *may not be the same). Hence it is regarded as a distribution in* V_I *(footnote 2 at p34, [3]), denoted by*

$$(\pi_I)_\star(\mathcal{T}_{I_1})$$

and called the projection of the de Rham distribution to V_I *with respect to the chart. The projection (with the* $\star$ *subscript) has an expression,*

$$\int_{(\pi_I)_\star(\mathcal{T}_{I_1})} f d\mu^I = \int_T \pi_I^*(f) d\mu^{I_1}. \tag{2.3}$$

for a test function $f \in \mathscr{D}(V_I)$.

2.1 Definition

Definition 2.2. *(Radon-Nikodym*) In the following, vectors or points in Euclidean space $\mathbb{R}^{\bullet}$ will be denoted by* **bold** *letters. Let $\mathbb{R}^r$ be the Euclidean space with the standard linear structure that has the basis $\mathbf{e}_1, \cdots, \mathbf{e}_r$ and coordinates $x = \{x_1, \cdots, x_r\}$. Let $d\mu_x$ be the Euclidean volume form*

$$dx_1 \wedge \cdots \wedge dx_r,$$

$$\boldsymbol{\lambda} = \lambda_1 \mathbf{e}_1 + \cdots + \lambda_r \mathbf{e}_r \in \mathbb{R}^r \tag{2.4}$$

be a varied vector in the open region such that $\lambda_i > 0, all\ i$. In the following we describe a particular type of path limits (iterated) of a function of $\boldsymbol{\lambda}$ as $\boldsymbol{\lambda} \to \mathbf{0}$. We divide the coordinates

$$\lambda_1, \cdots, \lambda_r$$

into groups as j_1 group , j_2 group, $\cdots$, j_l group such that

$$\mathbb{R}^r \simeq \mathbb{R}^{j_1} \oplus \mathbb{R}^{j_2} \oplus \cdots \oplus \mathbb{R}^{j_l}$$

where all indexes $j's$ are non-zero. It will be referred to as the

$$group \quad order. \tag{2.5}$$

Then we consider such $\boldsymbol{\lambda}$ that λ_i values in the same group are equal. We'll use the symbol $\lim\limits_{\boldsymbol{\lambda} \Rsh \mathbf{0}}$ to denote the particularly (ordered) iterated limit for $\boldsymbol{\lambda} \to \mathbf{0}$ (i.e. all $\lambda_{j_k} \to 0$) in the order

$$\lim_{\lambda_{j_l} \to 0} \cdots \lim_{\lambda_{j_1} \to 0}.$$

We name it as a zigzag limit. Let

$$\mathbf{u} = u_1 \mathbf{e}_1 + \cdots + u_r \mathbf{e}_r \in \mathbb{R}^r$$

be a point. Let $D_{\boldsymbol{\lambda}}$ be an invertible affine map in the form,

$$\begin{array}{ccc} \mathbb{R}^r & \to & \mathbb{R}^r \\ \mathbf{x} & \to & B \circ \mathbb{D}_{\boldsymbol{\lambda}}(\mathbf{x}) + \mathbf{u} \end{array} \tag{2.6}$$

referred to as the testing map, where B is an invertible linear map and $\mathbb{D}_{\boldsymbol{\lambda}}$ is the diagonal linear map

$$\begin{array}{ccc} \mathbb{R}^r & \to & \mathbb{R}^r \\ \mathbf{e}_i & \to & \lambda_i \mathbf{e}_i, all\ i. \end{array}$$

Denote the set of locally integrable functions by $\mathcal{L}^1_{loc}$. We say a bounded $L \in \mathcal{L}^1_{loc}(\mathbb{R}^r)$ is of Radon-Nikodym if for any test function $\phi \in \mathscr{D}(\mathbb{R}^r)$, any testing map $D_{\boldsymbol{\lambda}}$, any $\mathbf{u}$ and any group order, the zigzag limit

$$\lim_{\boldsymbol{\lambda} \Rsh \mathbf{0}} \int_{\mathbf{x} \in \mathbb{R}^r} L\Big(D_{\boldsymbol{\lambda}}(\mathbf{x})\Big) \phi(\mathbf{x}) d\mu_x \tag{2.7}$$

*Radon-Nikodym derivative is an important locally L^1 function in the theory of probability (see [1]), whose average values around the non a.e. points lie in the heart of convergence of (1.6).

exists. We denote the limit (2.4) by

$$RN_{\phi,L}, \tag{2.8}$$

and call it the Radon-Nikodym number.

Remark Zigzag limit is a particular type of path limits along continuous paths. However the function is not defined on the path.

Proposition 2.3. *It does not depend on coordinates for the bounded*

$$L \in \mathcal{L}^1_{loc}(\mathbb{R}^r)$$

to be of Radon-Nikodym.

Remark However the Radon-Nikodym number depends on coordinates. The invariance is due to the matrix B.

Proof. The proof for $\mathbf{u} \neq 0$ is identical with the homogeneous case where $\mathbf{u} = 0$. So let's prove the homogeneous case. Let $L \in \mathcal{L}^1_{loc}(\mathbb{R}^r)$ be bounded and of Radon-Nikodym in x-coordinates. Let $y = \{y_1, \cdots, y_r\}$ be another coordinates of $\mathbb{R}^r$, and

$$\begin{array}{ccc} \nu : \mathbb{R}^r & \to & \mathbb{R}^r \\ (x_1, \cdots, x_r) & \to & (y_1, \cdots, y_r) \end{array} \tag{2.9}$$

be the diffeomorphism between the x-y coordinates. So we assume the homogeneous case,

$$\nu(\mathbf{0}) = \mathbf{0}.$$

Denote the volume forms of $\mathbb{R}^r$ in y, x coordinates by $d\mu_y, d\mu_x$ respectively. So

$$d\mu_y = g(\mathbf{x}) d\mu_x,$$

where $g(\mathbf{x})$ is C^∞. Then the composition $L \circ \nu^{-1}$ denoted by L_y is also locally L^1. It is sufficient to show the convergence of the numbers

$$A_{\boldsymbol{\lambda}} = \int_{\mathbf{y} \in \mathbb{R}^r} L_y(D_{\boldsymbol{\lambda}}(\mathbf{y}))\phi(\mathbf{y}) d\mu_y \tag{2.10}$$

as $\boldsymbol{\lambda} \upharpoonright \mathbf{0}$, where $D_{\boldsymbol{\lambda}}$ is the testing map with the linear transformation B and $\phi(\mathbf{y}) \in \mathscr{D}(\mathbb{R}^r)$. First we use standard calculation to convert the expression to x-coordinates,

$$\begin{aligned} A_{\boldsymbol{\lambda}} &= \frac{1}{det(B)\prod_{i=1}^r \lambda_i} \int_{\mathbf{y} \in \mathbb{R}^r} L_y(\mathbf{y})\phi(D_{\boldsymbol{\lambda}}^{-1}(\mathbf{y})) d\mu_y && (2.11) \\ &= \frac{1}{det(B)\prod_{i=1}^r \lambda_i} \int_{\mathbf{x} \in \mathbb{R}^r} L_y(\nu(\mathbf{x}))\Big(\nu^*(\phi(D_{\boldsymbol{\lambda}}^{-1}(\mathbf{y})) d\mu_y)\Big) && (2.12) \\ &= \frac{1}{det(B)\prod_{i=1}^r \lambda_i} \int_{\mathbf{x} \in \mathbb{R}^r} L(\mathbf{x})\Big(\nu^*(\phi(D_{\boldsymbol{\lambda}}^{-1}(\mathbf{y})) d\mu_y)\Big) && (2.13) \end{aligned}$$

We make a change of variable

$$\mathbf{x} \Rightarrow B_0 \circ D_{\boldsymbol{\lambda}}(\mathbf{x})$$

(replacement of $\mathbf{x}$ with $B_0 \circ D_{\boldsymbol{\lambda}}(\mathbf{x})$) where $B_0 = \nu_*^{-1}|_{\mathbf{0}}$, a constant matrix under the y-basis. So $det(B_0)g(\mathbf{0}) = 1$. The integral in the last row (2.10) is

$$det(B_0) \int_{\mathbf{x}\in\mathbb{R}^r} L\big(B_0 \circ D_{\boldsymbol{\lambda}}(\mathbf{x})\big)\phi\big(D_{\boldsymbol{\lambda}}^{-1} \circ \nu \circ B_0 \circ D_{\boldsymbol{\lambda}}(\mathbf{x})\big)g\big(B_0 \circ D_{\boldsymbol{\lambda}}(\mathbf{x})\big)d\mu_x \quad (2.14)$$

Because ϕ has a compact support, the variable $\mathbf{x}$ in the integral (2.11) is bounded. Hence as $|\boldsymbol{\lambda}| \to 0$,

$$D_{\boldsymbol{\lambda}}^{-1} \circ \nu \circ B_0 \circ D_{\boldsymbol{\lambda}}(\mathbf{x})$$

uniformly (with respect to $\mathbf{x}$) converges to $\mathbf{x}$, and

$$B_0 \circ D_{\boldsymbol{\lambda}}(\mathbf{x})$$

to $\mathbf{0}$. Thus

$$\phi(D_{\boldsymbol{\lambda}}^{-1} \circ \nu \circ B_0 \circ D_{\boldsymbol{\lambda}}(\mathbf{x}))g(B_0 \circ D_{\boldsymbol{\lambda}}(\mathbf{x}))$$

uniformly converges to

$$\phi(\mathbf{x})g(\mathbf{0}).$$

Considering the limits in

$$\begin{aligned} A_{\boldsymbol{\lambda}} =& det(B_0) \int_{\mathbf{x}\in\mathbb{R}^r} L(B_0 \circ D_{\boldsymbol{\lambda}}(\mathbf{x})) \cdot \Big(\phi(D_{\boldsymbol{\lambda}}^{-1} \circ \nu \circ B_0 \circ D_{\boldsymbol{\lambda}}(\mathbf{x}))g(B_0 \circ D_{\boldsymbol{\lambda}}(\mathbf{x})) - \phi(\mathbf{x})g(\mathbf{0})\Big)d\mu_x \\ &+ det(B_0) \int_{\mathbf{x}\in\mathbb{R}^r} L(B_0 \circ D_{\boldsymbol{\lambda}}(\mathbf{x}))\phi(\mathbf{x})g(\mathbf{0})d\mu_x \end{aligned}$$

since the function

$$L(B_0 \circ D_{\boldsymbol{\lambda}}(\mathbf{x}))$$

is bounded, we conclude that

$$\lim_{\boldsymbol{\lambda}\to\mathbf{0}} A_{\boldsymbol{\lambda}} = \lim_{\boldsymbol{\lambda}\to\mathbf{0}} \int_{\mathbf{x}\in\mathbb{R}^r} L(B_0 \circ D_{\boldsymbol{\lambda}}(\mathbf{x}))\phi(\mathbf{x})d\mu_x.$$

Notice $B_0 \circ D_{\boldsymbol{\lambda}}$ is still a testing map. Hence

$$\lim_{\boldsymbol{\lambda}\to\mathbf{0}} \int_{\mathbf{x}\in\mathbb{R}^r} L(B_0 \circ D_{\boldsymbol{\lambda}}(\mathbf{x}))\phi(\mathbf{x})d\mu_x$$

converges. This completes the proof.

$\square$

Definition 2.4. *(Lebesgue current).*

Let $\mathcal{X}$ be a differentiable manifold of dimension m. Let U, a neighborhood, $x_1, \cdots, x_m$ coordinates for U be a chart in the differential structure of $\mathcal{X}$. Let $d\mu^I$ be the Euclidean volume form

$$dx_{i_1} \wedge \cdots \wedge dx_{i_r} \tag{2.15}$$

of an r-dimensional coordinates plane V_I with multi-index $I = (i_1 \cdots i_r)$,

$$\pi_I : U \to V_I \simeq \mathbb{R}^r$$

the projection given by the chart. Then a homogeneous current T of dimension p is called Lebesgue if for each chart $(U, x_1, \cdots, x_m)$ in an atlas and each set $\mathcal{S}$ of C^∞ forms $\xi \in \mathscr{D}(U)$ bounded to order 0, the following conditions are satisfied.

(a) Lebesgue condition

Let I, I_1 be any two multi-indexes with the same length. The projection $(\pi_I)_\star(\mathcal{T}_{I_1})$ of each de Rham distribution $\mathcal{T}_{I_1}$ of $T \wedge \xi$ to each coordinates plane V_I is a signed measure absolutely continuous with respect to the Lebesgue measure (defined by the chart). Furthermore the Radon-Nikodym derivative $\frac{d(\pi_I)_\star(\mathcal{T}_{I_1})}{d\mu^I}$ (see section 32, [1]) is a bounded L^1 function with the same compact support and for all ξ in set $\mathcal{S}$ of forms bounded to order 0 (see chapter III, §9 in [3]). This is equivalent to the existence of a Lebesgue integrable function $\mathcal{L}_I$ on V_I, that is bounded, supported in the same compact set and satisfies

$$\int_{(\pi_I)_\star(\mathcal{T}_{I_1})} \phi d\mu^I = \int_{V_I} \mathcal{L}_I \phi d\mu^I \tag{2.16}$$

for any test function $\phi \in \mathscr{D}(V_I)$. The L^1 function $\mathcal{L}_I = \frac{d(\pi_I)_\star(\mathcal{T}_{I_1})}{d\mu^I}$ will be called the Lebesgue function of T or $T \wedge \xi$. The formula (2.16) can be combined with (2.14) to have a more direct expression in terms of the original current T,

$$(2.16) = \int_{T \wedge \xi} (\pi_I)^*(\phi) d\mu^{I_1} \tag{2.17}$$

where the index I_1 is the index associated to the de Rham distribution $\mathcal{T}_{I_1}$, i.e.

$$T \wedge \xi = \mathcal{T}_{I_1} d\mu^{I_1^\diamond} + \cdots .$$

(note: index I is different from I_1, but has the same length).

(b) Radon-Nikodym condition.

All Lebesgue functions $\mathcal{L}_I$ of T are of Radon-Nikodym.

Remark Lebesgue functions of T are dependent of ξ and coordinates chart which are not reflected in the notation $\mathcal{L}_I$. It is a particular type of density functions in probability theory.† In integral theory it can be described as follows.

†A Radon-Nikodym derivative evaluated at an a.e. point is the infinitesimal ratio of two measures, called the density.

Proposition 2.5. *Assume all notations from Definition 2.4. Then Radon-Nikodym condition holds if and only if*

$$\lim_{\boldsymbol{\lambda}\to\mathbf{0}} \frac{1}{det(B)\prod_{i=1}^{r}\lambda_i}\int_{\mathbf{v}\in V_I}\mathcal{L}_I(\mathbf{v})\phi(D_{\boldsymbol{\lambda}}^{-1}(\mathbf{v}))d\mu^I \tag{2.18}$$

exists for each test function ϕ and index I. Furthermore if the Lebesgue function $\mathcal{L}_I$ is continuous at $\mathbf{u}$,

$$RN_{\phi,\mathcal{L}_I} = \mathcal{L}_I(\mathbf{u})\int_{\mathbf{v}\in V_I}\phi(\mathbf{v})d\mu^I. \tag{2.19}$$

Proof. Recall the integral (2.7) in Radon-Nikodym condition. We consider the integral

$$\int_{\mathbf{v}\in V_I}\mathcal{L}_I(D_{\boldsymbol{\lambda}}(\mathbf{v}))\phi(\mathbf{v})d\mu^I.$$

After the change of variables

$$D_{\boldsymbol{\lambda}}(\mathbf{v}) \Rightarrow \mathbf{v}. \tag{2.20}$$

(replacement of $D_{\boldsymbol{\lambda}}(\mathbf{v})$ with $\mathbf{v}$) the formula (2.7) turns to the formula (2.18). If $\mathcal{L}_I$ is continuous, since $\mathbf{v}$ is bounded, we have

$$\lim_{\boldsymbol{\lambda}\to\mathbf{0}}\int_{\mathbf{v}\in V_I}\mathcal{L}_I(D_{\boldsymbol{\lambda}}(\mathbf{v}))\phi(\mathbf{v})d\mu^I = \int_{\mathbf{v}\in V_I}\lim_{\boldsymbol{\lambda}\to\mathbf{0}}\mathcal{L}_I(D_{\boldsymbol{\lambda}}(\mathbf{v}))\phi(\mathbf{v})d\mu^I \tag{2.21}$$

Therefore the limit exists and is equal to

$$\int_{\mathbf{v}\in V_I}\mathcal{L}_I(\mathbf{u})\phi(\mathbf{v}))d\mu^I = \mathcal{L}_I(\mathbf{u})\int_{\mathbf{v}\in V_I}\phi(\mathbf{v})d\mu^I.$$

Thus the Radon-Nikodym condition is satisfied.

□

Definition 2.4 is stated in one atlas. Let's show it is independent of the atlas.

Proposition 2.6. *Definition 2.4 defines an invariant of the C^∞ differential structure.*

Proof. We need to prove that the conditions (a), (b) of Definition 2.4 are independent of charts. Let T be a current of dimension p, and $\xi \in \mathscr{D}(U)$ a form in a neighborhood U. Let $U, x = \{x_1, \cdots, x_m\}$ be a chart called x-chart satisfying the conditions of Definition 2.4 for $T \wedge \xi$. Let $U, y = \{y_1, \cdots, y_m\}$ be another chart called y-chart. Let ν be the transition map from x-chart to y-chart. Let

V_I be an r dimensional x coordinates plane, V_J be an r dimensional y coordinates plane, $d\mu_x^I, d\mu_y^J$ be the volume forms of the coordinates planes V_I, V_J respectively. Let

$$d\mu_y^{J^\diamond} = \sum_K g_{JK}(\mathbf{x}) d\mu_x^{K^\diamond} \tag{2.22}$$

where g_{JK} is the entry of the Jacobian matrix $J_{y\to x}$ from y-chart to x-chart, and K is a multi-index of length r. Let $\pi_J : U \to V_J$ be the projection through y-chart, and $\pi_I : U \to V_I$ the projection through the x-chart. We may assume the projection map (in y-chart)

$$\nu_{IJ} : V_I \quad \to \quad V_J$$

is a diffeomorphism that preserves the orientation. Now we fix J index of length r. On U, we have the sum

$$T \wedge \xi = \sum_K F_K(\mathbf{y}) d\mu_y^{K^\diamond} \tag{2.23}$$

where $F_K(\mathbf{y})$ is a de Rham distribution on U, and K is the multi-index of length r. Then $supp(F_K(\mathbf{y}))$ is bounded, since $T \wedge \xi$ has a compact support. Then for two fixed indexes J, K of length r,

$$F_K(\mathbf{y}) d\mu_y^{J^\diamond}$$

is a current on U of dimension r. Through x-chart it has the following decomposition

$$F_K(\mathbf{y}) d\mu_y^{J^\diamond} = \sum_I \mathcal{D}_I$$

where

$$\mathcal{D}_I = F_K(\nu(\mathbf{x})) g_{JI}(\mathbf{x}) d\mu_x^{I^\diamond} \tag{2.24}$$

is a current of dimension r on U, and $I^\diamond$ is a multi-index of length $m - r$. Note: $F_K(\nu(\mathbf{x}))$ is the distribution

$$(\nu^{-1})_*(F_K(\mathbf{y})).$$

This notation for push-forwards of distributions will be used alternately with the conventional notations throughout, but this is referred to as the change of variables.

There is a commutative diagram

$$\begin{tikzcd} & U \arrow[dl, "\pi_I"'] \arrow[dr, "\pi_J"] & \\ V_I \arrow[rr, "\nu_{IJ}"] & & V_J. \end{tikzcd} \tag{2.25}$$

Then we have

$$\begin{aligned}
&(\pi_J)_\star(F_K(\mathbf{y}))\\
&\text{(converted to a form)}\\
&=\sum_I (\pi_J)_*(\mathcal{D}_I)\\
&\text{(diagram (2.25))}\\
&=\sum_I (\nu_{IJ})_* \circ (\pi_I)_*(\mathcal{D}_I)
\end{aligned} \tag{2.26}$$

(Note: $\star$, $*$ are two different operators.). Therefore for distributions in y-chart we have

$$(\pi_J)_\star(F_K(\mathbf{y})) = \sum_I (\nu_{IJ})_* \circ (\pi_I)_*(\mathcal{D}_I). \tag{2.27}$$

Let's calculate $\mathcal{D}_I$. Let

$$\begin{aligned}
&T \wedge \xi = \sum_P G_P(\mathbf{x}) d\mu_x^{P^\diamond}\\
&\text{(Note: } G_P(\mathbf{x}) \text{ is some distribution)}\\
&= \sum_K \sum_P G_P(\nu^{-1}(\mathbf{y})) g_{PK}^{-1}(\mathbf{y}) d\mu_y^{K^\diamond},
\end{aligned}$$

where g_{PK}^{-1} stands for the entry of the Jacobian matrix, $J_{x\to y}$. Now we apply above calculation for

$$F_K(\mathbf{y}) = \sum_P G_P(\nu^{-1}(\mathbf{y})) g_{PK}^{-1}(\mathbf{y}),$$

and

$$\mathcal{D}_I = \sum_P G_P(\mathbf{x}) g_{PK}^{-1}(\nu(\mathbf{x})) g_{JI}(\mathbf{x}) d\mu_x^{I^\diamond}. \tag{2.28}$$

Since T is Lebesgue in x-chart, it satisfies both conditions of Definition 2.4 in x-chart, therefore $(\pi_I)_*(\mathcal{D}_I)$ is a distribution in x-chart. So it is is a bounded, compactly supported L^1 function of Radon-Nikodym on V_I in x-chart. Due to Proposition 2.2, so is

$$(\nu_{IJ})_\star \circ (\pi_I)_*(\mathcal{D}_I)$$

on V_J in y-chart. Hence its sum over finitely many I,

$$\sum_I (\nu_{IJ})_* \circ (\pi_I)_*(\mathcal{D}_I)$$

is also a bounded, compactly supported L^1 function of Radon-Nikodym on V_J (which is in y-chart). By the formula (2.27) we complete the proof.

□

Definition 2.7. *Let $\mathcal{X}$ be a C^∞ manifold. Denote the collection of Lebesgue currents by*

$$\mathscr{L}(\mathcal{X}).$$

2.2 Examples

It is clear that $\mathscr{L}(\mathcal{X})$ is a subspace. In this subsection we'll provide three major examples: 1) C^∞ singular chains; 2) C^∞ forms; 3) Cartesian product.

Theorem 2.8. *Let c be a regular cell. Then c is Lebesgue. Furthermore C^∞ chains are Lebesgue.*

The theorem is one of major theorems whose proof follows from the following two lemmas: 1) the proof of Lebesgue condition; 2) the proof of Randon-Nikodym condition.

- **Lebesgue condition**

Lemma 2.9. *A regular cell c satisfies Lebesgue condition.*

Proof. It suffices to work in one chart. So we assume $\mathcal{X} = U = \mathbb{R}^m$ is equipped with the standard chart (a basis for the linear space) with coordinates $(x_1, \cdots, x_m)$. We may assume the cell c is represented by a diffeomorphism extended to $\mathcal{K}$,

$$\begin{array}{rcl} h : \mathcal{K} & \to & h(\mathcal{K}) \subset U \\ \cup & & \cup \\ \Delta & \to & h(\Delta) \end{array} \tag{2.29}$$

where Δ is a polyhedron in an Euclidean space, and $\mathcal{K}$ is a neighborhood of Δ. Let ξ be a test form in $\mathscr{D}(U)$ such that

$$dim(\Delta) - deg(\xi) = r.$$

Let $V_I \simeq \mathbb{R}^r$ be an r-dimensional coordinates plane. We denote projection $U \to V_I$ by π_I. Let $d\mu^J$ be the Euclidean volume form of another r-dimensional coordinates plane. Then by the formula (2.17) the projection of a de Rham distribution of $c \wedge \xi$ to V_I is the functional,

$$\mathcal{F} : \phi \quad \to \quad \int_{c\wedge\xi} \phi(\mathbf{x}) d\mu^J \tag{2.30}$$

where $\phi(\mathbf{x}) = \pi_I^*(\phi(\mathbf{v}))$ for a test function $\phi(\mathbf{v}) \in \mathscr{D}^0(V_I)$ (Note: the index J is associated with the de Rham distribution). Notice that

$$\begin{array}{c} |\int_{c\wedge\xi} \phi(\mathbf{x}) d\mu^J| \\ = |\int_c \xi \wedge \phi(\mathbf{x}) d\mu^J| \\ = |\int_{\Delta^p} h^*(\xi \wedge \phi(\mathbf{x}) d\mu^J)| \\ \leq C||\phi||_{0,K} \end{array} \tag{2.31}$$

where C is a constant and $||\phi||_{0,K}$ is semi-norm with the compact support $K = supp(\phi(\mathbf{v}))$. Since the inequality (2.31) holds for all compact set K supporting the ϕ, by Proposition 2.1.8, 1.3.11, [2], $\mathcal{F}$ is a measure. So if we let ϕ be a characteristic function $\chi(E)$ of a subset set $E \subset V_I$ of Lebesgue measure 0, the inequality (2.31) becomes

$$|\int_{\mathcal{F}} \chi(E) d\mu^J| \leq C \int_{h^{-1}(E)} d\mu^J = 0. \tag{2.32}$$

Hence $\mathcal{F}$ is absolutely continuous with respect to the Lebesgue measure $d\mu^J$ of V_I. Next we estimate the Lebesgue function which is the Radon-Nikodym derivative a.e.

$$\lim_{\epsilon \to 0} \frac{|\int_{\Delta^p} h^*(\xi \wedge \chi(B_\epsilon) d\mu^J)|}{d\mu_I|_{\chi(B_\epsilon)}} \tag{2.33}$$

where B_ϵ is a bounded domain in V_I of radius ϵ, and $d\mu_I|_{\chi(B_\epsilon)}$ is its Lebesgue measure. Notice that

$$\left|\int_{\Delta^p} h^*(\xi \wedge \chi(B_\epsilon) d\mu^J)\right| \leq C||\xi||_{0,K} \int_{B_\epsilon} d\mu_I.$$

Therefore

$$\left|\frac{\int_{\Delta^p} h^*(\xi \wedge \chi(B_\epsilon) d\mu^J)}{\int_{B_\epsilon} d\mu_I}\right| \leq C||\xi||_{0,K}. \tag{2.34}$$

Hence the Radon-Nikodym derivative $\frac{d\mathcal{F}}{d\mu^I}$ is bounded when ξ is locally bounded. This shows c satisfies the Lebesgue condition. □

- **Radon-Nikodym condition**

First we state a technical definition. Let $\mathbb{R}^{k_1}$ be a subspace of $\mathbb{R}^r$ with a direct sum decomposition

$$\mathbb{R}^r = \mathbb{R}^{k_1} \oplus \mathbb{R}^{k_2}. \tag{2.35}$$

Let $W \subset \mathbb{R}^r$ be a bounded measurable set, and $\mathbf{a}$ a point on the boundary of W. We call an intersection $W \cap (B \times \mathbb{R}^{k_2})$ for a ball $B \subset \mathbb{R}^{k_1}$ an k_1-neighborhood.

Definition 2.10. *We say the domain $W \subset \mathbb{R}^r$ is a growing set along $\mathbb{R}^{k_1}$ at the center $\mathbf{a}$ if there is a k_1-neighborhood $U_{\mathbf{a}}$ containing $\mathbf{a}$ such that for any point $(\mathbf{b}_1, \mathbf{b}_2) \in U_{\mathbf{a}}$ there is real number $\epsilon > 0$, and an ϵ-line segment lying in W as follows*

$$L_\epsilon = \{\mathbf{a} + t(\mathbf{b}_1, 0) : \ 0 < t \leq \epsilon\} \subset W. \tag{2.36}$$

Lemma 2.11. *We continue the notation in Lemma 2.9. A regular cell satisfies the Radon-Nikodym condition.*

Proof. Let's now prove the Radon-Nikodym condition. We may assume $\mathbf{u} = \mathbf{0}$ and $B = identity$. Let $\phi(\mathbf{v})$ be the test function on V_I with $\mathbf{v} \in V_I$. Recall the projection $\pi_I : U \to V_I$, $D_{\boldsymbol{\lambda}}$ the testing map which is a block-wise scalar multiplication (see the formula (2.3)). Since being Lebesgue current is independent choice of coordinates, we may choose a coordinates system so that the composition

$$P : \mathcal{K} \to U \to V_I$$

is a diffieomorphism and $\mathcal{K} \simeq U \simeq \mathbb{R}^r$. Then by using Proposition 2.5, the Radon-Nikodym number is the limit

$$\lim_{\boldsymbol{\lambda} \Rsh 0} \int_{c \wedge \xi} \pi_I^* \left(\frac{\phi(D_{\boldsymbol{\lambda}}^{-1}(\mathbf{v}))}{det(\mathbb{D}_{\boldsymbol{\lambda}} B)} \right) d\mu^J, \tag{2.37}$$

where J is an arbitrary multi-index of length r, and $D_{\boldsymbol{\lambda}}(\bullet)$ is the affine transformation as in (2.3) (the integrand is the C^∞ form on $\mathbb{R}^r$). Because the integrand can absorb the C^∞ form ξ, so for the simplicity we may assume ξ has degree 0 and has value 1 on $\bar{c}$, and $I = J$ (note ξ is bounded by 1). Then after the change of variables

$$D_{\boldsymbol{\lambda}}^{-1}(\mathbf{v}) \Rightarrow \mathbf{v},$$

the integral in (2.37) is the evaluation of distributions on the plane $V_I \simeq \mathbb{R}^r$,

$$\int_{D_{\boldsymbol{\lambda}}^{-1}(P(\Delta))} \phi(\mathbf{v}) d\mu^J, \tag{2.38}$$

where $D_{\boldsymbol{\lambda}}^{-1}(P(\Delta))$ is a cell for each $\boldsymbol{\lambda}$, and $d\mu^J$ the Lebesgue measure.

Next we use measure theory to show

Claim 2.12. *The sequence of distributions*

$$D_{\boldsymbol{\lambda}}^{-1}(P(\Delta))$$

converges weakly as a zigzag limit $|\boldsymbol{\lambda}| \Rsh 0$.

Proof. of claim 2.12: First we consider the general situation for a bounded measurable set W as in the definition 2.10. Then

$$W = U_{\mathbf{a}} \cup U_{\mathbf{a}}^c \tag{2.39}$$

where $U_{\mathbf{a}} \subset W$ is a ball centered at the point $\mathbf{a}$, and $U_{\mathbf{a}}^c = W \backslash U_{\mathbf{a}}$. Let $\delta > 0$ be the radius of $U_{\mathbf{a}}$. Let $\mathcal{A}_\lambda$ for $\lambda > 0$ be the linear transformation

$$\mathbb{R}^r \quad \to \quad \mathbb{R}^r \tag{2.40}$$

represented by

$$\begin{pmatrix} \lambda I_{k_1} & 0 \\ 0 & I_{k_2} \end{pmatrix} \tag{2.41}$$

in the decomposition

$$\mathbb{R}^r = \mathbb{R}^{k_1} \oplus \mathbb{R}^{k_2},$$

where I_{k_i} are the identity matrix of size $k_i \times k_i$. Let $N \in \mathbb{N}$ be a natural number. Since $\delta > 0$, $\lim_{N\to\infty} \frac{\delta}{N} = 0$. Hence $\mathcal{A}_{\frac{1}{N}}(U^c_{\mathbf{a}})$ as a distribution converges to 0 weakly as $N \to \infty$. For the other set, since $U_{\mathbf{a}}$ is a growing set, we have a sequence of measurable sets

$$\mathcal{A}_1(U_{\mathbf{a}}) \subset \mathcal{A}_{\frac{N-1}{N}}(U_{\mathbf{a}}) \subset \cdots \subset \mathcal{A}_{\frac{1}{N}}(U_{\mathbf{a}}) \subset \cdots \tag{2.42}$$

Let $W_{k_1} = \cup_N \mathcal{A}_{\frac{1}{N}}(U_{\mathbf{a}})$. Then W_{k_1} is a measurable set and $\mathcal{A}_{\frac{1}{N}}(U_{\mathbf{a}})$ weakly converges to the measurable set W_{k_1}. Therefore $\mathcal{A}_{\frac{1}{N}}(W)$ as $N \to \infty$ converges to a measurable set W_{k_1}. Hence $\mathcal{A}_\lambda(W)$ as $\lambda \to 0$ converges to a measurable set W_{k_1}.

Then we repeat it for each division as follows. According to the group order of the zigzag limit, there is a decomposition

$$\mathbb{R}^r = \mathbb{R}^{j_1} \oplus \cdots \oplus \mathbb{R}^{j_l} \tag{2.43}$$

Since $P(\Delta)$ is a C^∞ r-cell in $\mathbb{R}^r$, its projection to each coordinate's plane is a growing set. So we can repeat above arguments for each block in the group order

$$j_1, j_2, \cdots, j_l.$$

Then we obtain each limit denoted by each of

$$\big(P(\Delta)\big)_{j_1}, \Big(\big(P(\Delta)\big)_{j_1}\Big)_{j_2}, \cdots, \bigg(\Big(\big(P(\Delta)\big)_{j_1}\Big)_{j_2} \cdots\bigg)_{j_l}$$

is a growing set in $\mathbb{R}^r$. Finally $D^{-1}_{\boldsymbol{\lambda}}(P(\Delta))$ converges weakly to the finite Lebesgue measurable set

$$\bigg(\Big(\big(P(\Delta)\big)_{j_1}\Big)_{j_2} \cdots\bigg)_{j_l}$$

as $|\boldsymbol{\lambda}| \Rsh 0$. We complete the proof of Claim 2.12. □

By the linearity, the existence is extended to all chains and cycles □

Theorem 2.8 follows from Lemmas 2.9, 2.11.

Proposition 2.13. *Let ω be a C^∞ form. Then ω is Lebesgue.*

Proof. We may prove it locally. So let $\mathcal{X} = \mathbb{R}^m$. Let $m - p = deg(\omega)$. Let ξ be any test form on $\mathbb{R}^m$ of degree $p - r$. Let

$$x_1, \cdots, x_r, x_{r+1}, \cdots, x_p, x_{p+1}, \cdots, x_m$$

be a coordinates chart. Let V_I have coordinates plane of components $x_1, \cdots, x_r$. For the simplicity, we may assume

$$\omega = M(\mathbf{x}) dx_{p+1} \wedge \cdots \wedge dx_m \tag{2.44}$$

$$\xi = N(\mathbf{x}) dx_{r+1} \wedge \cdots \wedge dx_p \tag{2.45}$$

We obtain that the Lebesgue function of $\omega \wedge \xi$ is the fibre integral

$$\int_{(x_{r+1}, \cdots, x_m) \in \mathbb{R}^{m-r}} M(\mathbf{x}) N(\mathbf{x}) dx_{p+1} \wedge \cdots \wedge dx_m \wedge dx_{r+1} \wedge \cdots \wedge dx_p \tag{2.46}$$

which is a C^∞ function of $x_1, \cdots, x_r$ in the V_I plane. Since the Lebesgue function is C^∞, the Radon-Nikodym condition is satisfied □

Next we work with Cartesian product.

Lemma 2.14. *We resume the set-up of Definition 2.4. In particular U is a chart of the manifold $\mathcal{X}$. Let $\mathcal{L}_I$ be a bounded L^1 function in $\mathcal{L}^1_{loc}(U)$, where $\mathcal{L}^1_{loc}(\cdot)$ denotes the set of locally integrable functions. If*

$$\phi \in \mathscr{D}(U \times \mathbb{R}^k), \tag{2.47}$$

(1) then Radon-Nikodym number $RN_{\phi, \mathcal{L}_I}(\mathbf{y})$, which is a function of $\mathbf{y} \in \mathbb{R}^k$ lies in $\mathscr{D}(\mathbb{R}^k)$.

(2) The convergence

$$\lim_{\boldsymbol{\lambda} \vec{\to} 0} \int_{\mathbf{v} \in V_I} \mathcal{L}_I(D_{\boldsymbol{\lambda}}(\mathbf{v})) \phi(\mathbf{v}, \mathbf{y}) d\mu^I \tag{2.48}$$

is uniform for the bounded variable $\mathbf{y} \in \mathbb{R}^k$.

Proof. (1) Let $\mathbf{e}_i, i = 1, \cdots, n$ be a basis for $\mathbb{R}^k$. Let h be a real number,

$$\mathbf{y} = \sum_{i=1}^{n} y_i \mathbf{e}_i \in \mathbb{R}^k.$$

Let's consider the number

$$\begin{aligned} A_h &= \frac{RN_{\phi, \mathcal{L}_I}(\mathbf{y} + h\mathbf{e}_i) - RN_{\phi, \mathcal{L}_I}(\mathbf{y})}{h} - RN_{\frac{\partial \phi(\mathbf{y})}{\partial y_i}, \mathcal{L}_I}(\mathbf{y}) \\ &= \lim_{|\boldsymbol{\lambda}| \vec{\to} \mathbf{0}} \int_{\mathbf{x} \in V_I} \mathcal{L}_I(D_{\boldsymbol{\lambda}}(\mathbf{x})) \left(\frac{\phi\big(D_{\boldsymbol{\lambda}}(\mathbf{x}), \mathbf{y} + \Delta y \mathbf{e}_i\big) - \phi\big(D_{\boldsymbol{\lambda}}(\mathbf{x}), \mathbf{y}\big)}{h} - \frac{\partial \phi\big(D_{\boldsymbol{\lambda}}(\mathbf{x}), \mathbf{y}\big)}{\partial y_i} \right) d\mu^I \end{aligned}$$

Since ϕ is C^∞ with a compact support,

$$\frac{\phi\big(D_{\boldsymbol\lambda}(\mathbf{x}), \mathbf{y}+h\mathbf{e}_i\big)-\phi\big(D_{\boldsymbol\lambda}(\mathbf{x}), \mathbf{y}\big)}{h}-\frac{\partial\phi\big(D_{\boldsymbol\lambda}(\mathbf{x}), \mathbf{y}\big)}{\partial y_i}$$

as $h \to 0$ uniformly (with respect to $\boldsymbol\lambda, \mathbf{x}$) converges to 0. Together with the bounded $\mathcal{L}_I(D_{\boldsymbol\lambda}(\mathbf{x}))$, we have

$$\lim_{\Delta y\to\mathbf{0}} A_h = 0.$$

Hence $RN_{\phi,\mathcal{L}_I}(\mathbf{y})$ is differentiable and

$$\frac{\partial RN_{\phi,\mathcal{L}_I}(\mathbf{y})}{\partial y_i} = RN_{\frac{\partial\phi(\mathbf{y})}{\partial y_i},\mathcal{L}_I}(\mathbf{y}) \tag{2.49}$$

By the iteration, $RN_{\phi,\mathcal{L}_I}(\mathbf{y})$ is C^∞. Since $\phi(\mathbf{x},\mathbf{y})$ is both bounded and compactly supported, so is $RN_{\phi,\mathcal{L}_I}(\mathbf{y})$.

(2) Let's continue from part (1). By Theorem 6, Chapter II, §7, [3], there is a sequence of test functions

$$\psi_1^n(\mathbf{v}) \in \mathscr{D}(V_I), \psi_2^n(\mathbf{y}) \in \mathscr{D}(\mathbb{R}^k) \tag{2.50}$$

such that

$$\psi_1^n(\mathbf{v})\psi_2^n(\mathbf{y}) \to \phi(\mathbf{v},\mathbf{y}) \; as \; n\to\infty$$

uniformly on the compact set. Thus for any $\epsilon > 0$, since $\mathcal{L}_I$ is bounded there is an N such that

$$\Big|\int_{\mathbf{v}\in V_I} \mathcal{L}_I(D_{\boldsymbol\lambda}(\mathbf{v}))\Big(\psi_1^N(\mathbf{v})\psi_2^N(\mathbf{y}) - \phi(\mathbf{v},\mathbf{y})\Big)d\mu^I\Big| \le \epsilon \tag{2.51}$$

for all $\boldsymbol\lambda$. Taking the limit $|\boldsymbol\lambda| \Rsh 0$, we have inequality

$$|\psi_2^N(\mathbf{y})RN_{\psi_1^N,\mathcal{L}_I} - RN_{\phi,\mathcal{L}_I}(\mathbf{y})| \le \epsilon \tag{2.52}$$

(which does not involve $\boldsymbol\lambda$). Next we write the number $\psi_2^N(\mathbf{y})RN_{\psi_1^N,\mathcal{L}_I}$ as a zigzag limit $|\boldsymbol\lambda| \Rsh 0$:

$$\psi_2^N(\mathbf{y})\int_{\mathbf{v}\in V_I}\mathcal{L}_I(D_{\boldsymbol\lambda}(\mathbf{v}))\psi_1^N(\mathbf{v})d\mu^I \quad\longrightarrow\quad \psi_2^N(\mathbf{y})RN_{\psi_1^N,\mathcal{L}_I} \tag{2.53}$$

whose convergence is independent of $\mathbf{y}$. Hence the convergence

$$\int_{\mathbf{v}\in V_I}\mathcal{L}_I(D_{\boldsymbol\lambda}(\mathbf{v})\phi(\mathbf{v},\mathbf{y})d\mu^I \to RN_{\phi,\mathcal{L}_I} \tag{2.54}$$

as $|\boldsymbol\lambda| \Rsh \mathbf{0}$ is independent of $\mathbf{y}$.

$\square$

Theorem 2.15. *Let $\mathcal{Y}$ be another C^∞ manifold. If currents T_1, T_2 are Lebesgue in $\mathcal{X}, \mathcal{Y}$ respectively, so is $T_1 \times T_2$ in $\mathcal{X} \times \mathcal{Y}$, where $T_1 \times T_2$ is the wedge product deduced from the tensor product of two currents.*

Proof of Theorem 2.15. There are 3 steps.
(1) *SETUP*. By Proposition 2.6, it suffices to work with one chart. So we assume $\mathcal{X} = \mathbb{R}^m$ whose points are denoted by $\mathbf{x}$ and objects are labeled by x. We also assume $\mathcal{Y} = \mathbb{R}^n$ whose points are denoted by $\mathbf{y}$ and objects are labeled by y. For the clarity, we'll use the indexes in the following convention.

(I) Single indexes denote objects from each individual manifold $\mathcal{X}$ or $\mathcal{Y}$. Indexes p, k with $p \geq k$ denote the objects in $\mathcal{X}$, q, l with $q \geq l$ in $\mathcal{Y}$.
(II) Double indexes denote the objects from the product $\mathcal{X} \times \mathcal{Y}$.
(III) $V_\bullet$, $V_{\bullet,\bullet}$ are subspaces, and $d\mu_\bullet, d\mu_{\bullet,\bullet}$ are the Lebesgue measures for subspaces inherited from the fixed Lebesgue measures on $\mathcal{X}, \mathcal{Y}$.

Recall T_1, T_2 are currents. Let's assume $dim(T_1) = p, dim(T_2) = q$. We may assume the form ξ is in the format

$$\xi = \zeta(\mathbf{x}, \mathbf{y}) d\mu_{p-k,q-l} \tag{2.55}$$

with the function $\zeta \in \mathscr{D}(\mathcal{X} \times \mathcal{Y})$ where $d\mu_{p-k,q-l}$ is the Lebesgue measure for some subspaces $V_{p-k} \times V_{q-l} \subset \mathcal{X} \times \mathcal{Y}$. Let $\xi_x \in \mathscr{D}(\mathcal{X}), \xi_y \in \mathscr{D}(\mathcal{Y})$ be functions such that they are equal to 1 on the projections of $supp(\zeta)$ to $\mathcal{X}, \mathcal{Y}$. We denote $\xi_x T_1$ by T_x and $\xi_y T_2$ by T_y. They all have compact supports. Then

$$(T_1 \times T_2) \wedge \xi = (T_x d\mu_{p-k} \times T_y d\mu_{q-l}) \zeta(\mathbf{x}, \mathbf{y}) \tag{2.56}$$

where

$$T_x d\mu_{p-k} (resp.\ T_y d\mu_{q-l})$$

is the abbreviation for

$$T_x \wedge d\mu_{p-k} (resp.\ T_y \wedge d\mu_{q-l}).$$

Let

$$\pi_{k,l} : \mathcal{X} \times \mathcal{Y} \quad \to \quad V_k \times V_l \tag{2.57}$$

be the projection. Let $d\mu_x, d\mu_y$ be the Euclidean volume forms of other coordinates planes of dimensions k, l in $\mathcal{X}, \mathcal{Y}$ respectively. By "other", it means the the subspaces may not be the same as V_k, V_l. Then by the formula (2.17), the projection of a de Rham distribution of current $(T_1 \times T_2) \wedge \xi$ to $V_{k,l} = V_k \times V_l$ is defined to be the functional

$$\begin{aligned} \mathcal{F} : \phi \quad \to \quad & \int_{(T_x d\mu_{p-k} \times T_y d\mu_{q-l}) \wedge \zeta(\mathbf{x}, \mathbf{y})} \pi_{k,l}^*(\phi(\mathbf{x}_k, \mathbf{y}_l)) d\mu_x \wedge d\mu_y \\ = & \int_{T_y d\mu_{q-l}} \left(\int_{(T_x d\mu_{p-k}) \wedge \zeta(\mathbf{x}, \mathbf{y})} \pi_{k,l}^*(\phi(\mathbf{x}_k, \mathbf{y}_l)) d\mu_x \right) d\mu_y. \end{aligned} \tag{2.58}$$

where $\phi(\mathbf{x}_k, \mathbf{y}_l)$ is a test function on the coordinates plane $V_{k,l}$, iteration is well-defined for the compactly supported currents on C^∞ forms. In the following we'll address the properties of the distribution $\mathcal{F}$.

(2) *LEBESGUE CONDITION*.

By Theorem 6, Chapter II, §7, [3], there are sequences of test functions

$$\zeta_x^N(\mathbf{x}), \zeta_y^N(\mathbf{y}), N \in \mathbb{N} \tag{2.59}$$

on $\mathcal{X}, \mathcal{Y}$ respectively such that they are bounded for all variables $N, \mathbf{x}, \mathbf{y}$ and

$$\zeta_x^N(\mathbf{x})\zeta_y^N(\mathbf{y}) \to \zeta(\mathbf{x}, \mathbf{y}) \ uniformly, \ as \ N \to \infty$$

Then for any natural number N, we rewrite

$$\begin{aligned} \int_{\mathcal{F}} \phi d\mu_x \wedge d\mu_y = & \int_{T_y d\mu_{q-l}} \left(\int_{(T_x d\mu_{p-k}) \wedge \left(\zeta(\mathbf{x},\mathbf{y}) - \zeta_x^N(\mathbf{x})\zeta_y^N(\mathbf{y})\right)} \pi_{k,l}^*(\phi(\mathbf{x}_k, \mathbf{y}_l)) d\mu_x \right) d\mu_y \\ & + \int_{T_y d\mu_{q-l} \wedge \zeta_y^N(\mathbf{y})} \left(\int_{T_x d\mu_{p-k} \wedge \zeta_x^N(\mathbf{x})} \pi_{k,l}^*(\phi(\mathbf{x}_k, \mathbf{y}_l)) d\mu_x \right) d\mu_y. \end{aligned} \tag{2.60}$$

Now we let $\mathcal{L}_k^N(\mathbf{x}_k), \mathcal{L}_l^N(\mathbf{y}_l)$ be the Lebesgue functions of de Rham distributions of the currents

$$T_x d\mu_{p-k} \wedge \zeta_x^N(\mathbf{x}), \qquad T_y d\mu_{q-l} \wedge \zeta_y^N(\mathbf{y})$$

on V_k, V_l respectively. The Lebesgue condition implies they are bounded for all N, i.e. there is constant M such that

$$\begin{aligned} |\mathcal{L}_k^N(\mathbf{x}_k)| \leq M \\ |\mathcal{L}_l^N(\mathbf{y}_l)| \leq M \end{aligned} \tag{2.61}$$

for all N and bounded $\mathbf{x}_k, \mathbf{y}_l$ a.e. By part (1) of Lemma 2.14, the second term of (2.60) above continues to be

$$\begin{aligned} & \int_{T_y d\mu_{q-l} \wedge \zeta_y^N(\mathbf{y})} \left(\int_{T_x d\mu_{p-k} \wedge \zeta_x^N(\mathbf{x})} \phi(\mathbf{x}_k, \mathbf{y}_l) d\mu_x \right) d\mu_y \\ & = \int_{V_{k,l}} \mathcal{L}_k^N(\mathbf{x}_k) \mathcal{L}_l^N(\mathbf{y}_l) \pi_{k,l}^*(\phi(\mathbf{x}_k, \mathbf{y}_l)) d\mu_k \wedge d\mu_l \qquad . \end{aligned}$$

On the other hand, for the first term of (2.60), there is a sequence of numbers $a_N \to +\infty$ as $N \to +\infty$ such that the set of forms

$$a_N((\zeta(\mathbf{x}, \mathbf{y}) - \zeta_x^N(\mathbf{x}) \wedge \zeta_y^N(\mathbf{y}))$$

for all $N \in \mathbb{N}$ is locally bounded. Hence the Lebesgue functions of two currents

$$a_N(T_x d\mu_{p-k}) \wedge \left(\zeta(\mathbf{x}, \mathbf{y}) - \zeta_x^N(\mathbf{x}) \wedge \zeta_y^N(\mathbf{y})\right), \quad T_y d\mu_{q-l}$$

on V_k, V_l are bounded for all N (the Lebesgue function on V_k is dependent of $\mathbf{y}$, but it also bounded for all $\mathbf{y}$.). So the sequence of numbers

$$a_N \int_{T_y d\mu_{q-l}} \left(\int_{(T_x d\mu_{p-k}) \wedge \left(\zeta(\mathbf{x},\mathbf{y}) - \zeta_x^N(\mathbf{x}) \wedge \zeta_y^N(\mathbf{y})\right)} \pi_{k,l}^*(\phi(\mathbf{x}_k, \mathbf{y}_l)) d\mu_x \right) d\mu_y$$

for all N is bounded. Thus the sequence of real numbers

$$\int_{T_y d\mu_{q-l}} \left(\int_{(T_x d\mu_{p-k})\wedge\left(\zeta(\mathbf{x},\mathbf{y})-\zeta_x^N(\mathbf{x})\wedge\zeta_y^N(\mathbf{y})\right)} \pi_{k,l}^*(\phi(\mathbf{x}_k,\mathbf{y}_l))d\mu_x \right) d\mu_y$$

converges to 0 as $N \to \infty$. Therefore

$$\int_{\mathcal{F}} \phi d\mu_x \wedge d\mu_y = \lim_{N\to\infty} \int_{(\mathbf{x}_k,\mathbf{y}_l)\in V_{k,l}} \mathcal{L}_k^N(\mathbf{x}_k)\mathcal{L}_l^N(\mathbf{y}_l)\pi_{k,l}^*(\phi(\mathbf{x}_k,\mathbf{y}_l))d\mu_k \wedge d\mu_l.$$

(the Lebesgue integral exists due to the part (1) of Lemma 2.14). Then we apply the Lebesgue integral to estimate

$$\left| \int_{\mathcal{F}} \phi d\mu_x \wedge d\mu_y \right| \le C||\phi||_{0,K} \tag{2.62}$$

for some constant C. By Proposition 2.1.8, 1.3.11, [2], $\mathcal{F}$ is a distribution of order 0, thus a measure. If χ is a characteristic function of a set with 0 Lebesgue measure, the inequality (2.62) implies

$$\int_{\mathcal{F}} \chi d\mu_x \wedge d\mu_y = 0.$$

Thus $\mathcal{F}$ is a measure absolutely continuous with respect to the Lebesgue measure. The Lebesgue integral also shows that the Radon-Nikodym derivative has inequality

$$\left| \frac{d\mathcal{F}}{d\mu_{k,l}} \right| \le C' M^2 \tag{2.63}$$

for some constant C', where $d\mu_{k,l}$ is the Lebesgue measure for $V_{k,l}$ and the bound M is from (2.61). We complete the proof of the Lebesgue condition.

(3) $RADON - NIKODYM\ CONDITON$. Next we prove the Radon-Nikodym condition. Let

$$\mathcal{L}_{k,l}(\mathbf{x}_k,\mathbf{y}_l) := \frac{d\mathcal{F}}{d\mu_{k,l}}$$

be the Lebesgue function, where $d\mu_{k,l}$ is the Lebesgue measure of the plane $V_{k,l}$. Let $D_{\boldsymbol{\lambda}_1}, D_{\boldsymbol{\lambda}_2}$ are testing maps for Euclidean spaces V_k, V_l as in (2.3). Let $D_{(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)}$ be the testing map for $V_{k,l}$. Denote its identity extension to $\mathbb{R}^m \times \mathbb{R}^n$ by the same notation $D_{(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)}$. Let

$$C_{(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)}^N = \int_{(\mathbf{x}_k,\mathbf{y}_l)\in V_{k,l}} \mathcal{L}_k^N(D_{\boldsymbol{\lambda}_1}(\mathbf{x}_k))\mathcal{L}_l^N(D_{\boldsymbol{\lambda}_2}(\mathbf{y}_l))\pi_{k,l}^*(\phi(\mathbf{x}_k,\mathbf{y}_l))d\mu_k \wedge d\mu_l. \tag{2.64}$$

Then it is sufficient to prove the zigzag convergence of

$$\lim_{N\to\infty} C_{(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)}^N \tag{2.65}$$

as $(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) \Rsh \mathbf{0}$, i.e. the convergence of the iterated limit

$$\lim_{(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)\Rsh\mathbf{0}} \lim_{N\to\infty} C^N_{(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)}. \tag{2.66}$$

So we consider the other order

$$\lim_{N\to\infty} \lim_{(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)\Rsh\mathbf{0}} C^N_{(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)}.$$

Let

$$R_N = \lim_{(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)\Rsh\mathbf{0}} C^N_{(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)}. \tag{2.67}$$

Using Lemma 2.14 for the iterated evaluation, we see R_N exists and is bounded for all N.

Claim 2.16. *The sequence*

$$R_{N_1} - R_{N_2}, \qquad N_i \in \mathbb{N}.$$

converges to 0 *as* $(N_1, N_2) \to (\infty, \infty)$.

Proof of Claim 2.16. Let $a_{(N_1,N_2)}$ be a sequence of real numbers such that

$$\lim_{(N_1,N_2)\to(\infty,\infty)} a_{(N_1,N_2)} = \infty$$

and

$$a_{(N_1,N_2)}\big(\zeta_x^{N_1}(\mathbf{x}) \wedge \zeta_y^{N_1}(\mathbf{y}) - \zeta_x^{N_2}(\mathbf{x}) \wedge \zeta_y^{N_2}(\mathbf{y})\big)$$

is bounded for all N_1, N_2. Then

$$\begin{aligned} &a_{(N_1,N_2)}(R_{N_1} - R_{N_2}) \\ &= \lim_{(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)\Rsh\mathbf{0}} \int_{(T_x d\mu_{p-k}\times T_y d\mu_{q-l})\wedge a_{(N_1,N_2)}\left(\zeta_x^{N_1}\wedge(\mathbf{x})\zeta_y^{N_1}(\mathbf{y}) - \zeta_x^{N_2}(\mathbf{x})\wedge\zeta_y^{N_2}(\mathbf{y})\right)} \eta_{(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)} \end{aligned}$$

where $\eta_{(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)}$ is the C^∞ form

$$D^*_{(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)}(\pi^*_{k,l}(\phi(\mathbf{x}_k, \mathbf{y}_l))d\mu_k \wedge d\mu_l).$$

Let $\mathcal{J}_{(N_1,N_2)}$ be a Lebesgue function of the current

$$\left(T_x d\mu_{p-k} \times T_y d\mu_{q-l}\right)\left(a_{(N_1,N_2)}\big(\zeta_x^{N_1}(\mathbf{x}) \wedge \zeta_y^{N_1}(\mathbf{y}) - \zeta_x^{N_2}(\mathbf{x}) \wedge \zeta_y^{N_2}(\mathbf{y})\big)\right)$$

on $V_{k,l}$. Since

$$a_{(N_1,N_2)}\big(\zeta_x^{N_1}(\mathbf{x}) \wedge \zeta_y^{N_1}(\mathbf{y}) - \zeta_x^{N_2}(\mathbf{x}) \wedge \zeta_y^{N_2}(\mathbf{y})\big)$$

are C^∞ forms bounded locally, by step 2 above $\mathcal{J}_{(N_1,N_2)}$ is a bounded L^1 form. Thus

$$\begin{aligned}
&|\int_{(\mathbf{x}_k,\mathbf{y}_l)\in V_{k,l}} \mathcal{J}_{(N_1,N_2)}(D_{(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)}(\mathbf{x}_k,\mathbf{y}_l))\pi^*_{k,l}(\phi(\mathbf{x}_k,\mathbf{y}_l))d\mu_k\wedge d\mu_l| \\
&= a_{(N_1,N_2)}\lim_{(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)\to\mathbf{0}}|\int_{\left(T_x d\mu_{p-k}\times T_y d\mu_{q-l}\right)\wedge\left(\zeta_x^{N_1}(\mathbf{x})\wedge\zeta_y^{N_1}(\mathbf{y})-\zeta_x^{N_2}(\mathbf{x})\wedge\zeta_y^{N_2}(\mathbf{y})\right)}\eta_{(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)}| \\
&\le M''
\end{aligned}$$

for a positive number M''. Therefore

$$a_{(N_1,N_2)}(R_{N_1}-R_{N_2})$$

is bounded. Notice

$$\lim_{(N_1,N_2)\to(\infty,\infty)} a_{(N_1,N_2)}=\infty.$$

Thus

$$\lim_{(N_1,N_2)\to(\infty,\infty)}(R_{N_1}-R_{N_2})=0.$$

So R_N converges to a real number L. This shows the limit

$$\lim_{N\to\infty}\lim_{(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)\to\mathbf{0}} C^N_{(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)}=L$$

exists.

Now by the same proof the convergence above, the convergence of another limit

$$\lim_{N\to\infty} C^N_{(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)} \tag{2.68}$$

is uniformly independent of $\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2$. Hence the iterated limit in the opposite order,

$$\lim_{(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)\to\mathbf{0}}\lim_{N\to\infty} C^N_{(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)}$$

exists and is equal to the limit,

$$\lim_{N\to\infty}\lim_{(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)\to\mathbf{0}} C^N_{(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)}=L. \tag{2.69}$$

We complete the proof.

□

Proposition 2.17. *If T is Lebesgue and ω is C^∞, then the intersection*

$$T\wedge\omega \tag{2.70}$$

is Lebesgue.

Proof. This is the tautology. Let $\xi \in \mathscr{D}(U)$. Notice $\omega \wedge \xi \in \mathscr{D}(U)$. Then the projection $\mathcal{J}$ of de Rham distributions of

$$(T \wedge \omega) \wedge \xi$$

is the same as that of

$$T \wedge (\omega \wedge \xi).$$

We complete the proof. □

Example 2.18. *There exist currents that are not Lebesgue.*

In the Euclidean space $\mathbb{R}^m$ of coordinates $x_1, \cdots, x_p, \cdots, x_m$, we let

$$T = \delta_{\mathbf{0}} dx_{p+1} \wedge \cdots \wedge dx_m$$

with δ-function $\delta_{\mathbf{0}}$ of the origin $\mathbf{0}$ of $\mathbb{R}^m$. Let V be the coordinates plane with coordinates $x_1, \cdots, x_p$, and $\pi : \mathbb{R}^m \to V$ be the projection. Let $\xi \in \mathscr{D}(\mathbb{R}^m)$ with $\xi(\mathbf{0}) \neq 0$. So a projection of the de Rham distribution $\pi_\star(\xi\delta_{\mathbf{0}})$ is equal to

$$\delta_{\mathbf{0}}\xi(\mathbf{0}). \tag{2.71}$$

Hence $\pi_\star(\xi\delta_{\mathbf{0}})$ is the distribution $\delta_{\mathbf{0}}\xi(\mathbf{0})$, also a measure on V with the Borel σ-algebra of V. Now we consider the two measures for V on the same σ-algebra. When they are applied to the singleton set, the origin of V, the Lebesgue measure is 0, but the projection measure is $\xi(\mathbf{0}) \neq 0$. Hence

$$\pi_\star(\xi\delta_{\mathbf{0}}) \not\ll \text{ Lebesgue measure.}$$

So T does not satisfy the Lebesgue condition.

3 De Rham's Regularization

G. de Rham introduced the notion of currents that connects the singular chains and C^∞ forms. The connection is through the de Rham's regularization which consists of two operators: R_ϵ, A_ϵ (see chapter III, [3]). They are the original parts of de Rham theory which serves as the foundation to the differential geometry. However since we need to go beyond them, let's have a review.

3.1 Construction

Definition 3.1. *Let $\mathcal{X}$ be a connected, oriented manifold. Let ϵ be a small positive number. Linear operators R_ϵ and A_ϵ on $\mathscr{D}'(\mathcal{X})$ are called de Rham's regulator and homotopy operator respectively if they satisfy*

(1) a homotopy formula

$$R_\epsilon T - T = bA_\epsilon T + A_\epsilon bT. \tag{3.1}$$

where b is the boundary operator.

(2) $supp(R_\epsilon T), supp(A_\epsilon T)$ are contained in any given neighborhood of $supp(T)$ provided ϵ is sufficiently small.

(3) $R_\epsilon T$ is C^∞;

(4) If T is C^r, $A_\epsilon T$ is C^r.

(5) If a smooth differential form ϕ varies in a bounded set and ϵ is bounded above, then $R_\epsilon\phi, A_\epsilon\phi$ are bounded.

(6)

$$\lim_{\epsilon\to 0} R_\epsilon T = T$$

in the weak topology of $\mathscr{D}'(X)$.[‡]

Theorem 3.2. *(G. de Rham) The operators R_ϵ, A_ϵ exist.*

Proof. In the following we review the constructions of operators R_ϵ and A_ϵ. The verification of conditions (1)-(6) in [3] will be omitted. There are three steps in the construction.

Step 1: Local construction, i.e. the construction in $\mathcal{X} = \mathbb{R}^m$.

Step 2: Preparation. To prepare for the gluing, we "shrink" the operators to a bounded domain B in $\mathbb{R}^m$ with boundary.

Step 3: Gluing. Assume $\mathcal{X}$ is covered by the bounded domain with boundary B^i, *countable* i. Then glue the operators in each B^i to obtain the global

$$R_\epsilon, A_\epsilon \tag{3.2}$$

Step 1: The most part of this step is originated from Schwartz's work in [4]. But we'll explore it a little further. Let $\mathcal{X} = \mathbb{R}^m$ be the Euclidean space of dimension m with the standard linear structure. Let $x = (x_1, \cdots, x_m)$ be its Euclidean coordinates, and vectors and points in $\mathbb{R}^m$ will be denoted by

[‡]De Rham's convergence in [3] is a little stronger than the weak convergence. But no matter how strong the convergence is, the non-triviality lies ahead.

the **bold** letters. Let T be a homogeneous current of degree p on $\mathbb{R}^m$. Let $f(\mathbf{x}) \in \mathscr{D}(X)$ satisfying

$$\int_{\mathbf{x}\in\mathbb{R}^m} f(\mathbf{x}) d\mu_x = 1, \tag{3.3}$$

where $d\mu_x$ is the Euclidean volume form

$$dx_1 \wedge \cdots \wedge dx_m.$$

We assume f is symmetric,[§] i.e. $f(\mathbf{x}) = f(-\mathbf{x})$ Let

$$\vartheta_1(\mathbf{x}) = f(\mathbf{x}) d\mu_x. \tag{3.4}$$

Denote

$$\vartheta_\epsilon(\mathbf{x}) = \vartheta_1(\frac{\mathbf{x}}{\epsilon}). \tag{3.5}$$

be the m-form on $\mathbb{R}^m$.

Next we define two operators on the differential forms of Euclidean space $\mathbb{R}^m$ based on C^∞ maps $s_{\mathbf{y}}(\mathbf{x})$ below. Let

$$s_{\mathbf{y}}(\mathbf{x})$$

be C^∞ maps parametrized by $\mathbf{y} \in \mathbb{R}^m$,

$$\begin{array}{ccc} \mathbb{R}^m & \to & \mathbb{R}^m \\ \mathbf{x} & \to & s_{\mathbf{y}}(\mathbf{x}) \end{array}$$

such that all partial derivatives of the components with respect to the variables of $\mathbf{x}$ are continuous functions in $(\mathbf{x}, \mathbf{y})$. Let ϕ be a test form on $\mathbb{R}^m$. For such maps $s_{\mathbf{y}}(\mathbf{x})$, we denote two operations on the form ϕ

$$s^*_{\mathbf{y}}(\phi), \qquad \textit{and}$$

$$\mathbf{S}^*_{\mathbf{y}}(\phi) = Proj_*(s^*_{(t,\mathbf{y})}(\phi)), t \in [0,1]$$

where $Proj : [0,1] \times \mathcal{X} \to \mathcal{X}$ is the projection, $Proj_*$ is its fibre integral, and

$$\begin{array}{rcl} s_{(t,\mathbf{y})} : [0,1] \times \mathcal{X} & \to & \mathcal{X} \\ (t, x) & \to & s_{t\mathbf{y}}(\mathbf{x}). \end{array}$$

Then we define operators R_ϵ, A_ϵ on currents T by

$$\begin{cases} \int_{R_\epsilon T} \phi = \int_{\mathbf{x}\in T} \left(\int_{\mathbf{y}\in\mathbb{R}^m} \vartheta_\epsilon(\mathbf{y}) \wedge s^*_{\mathbf{y}} \phi(\mathbf{x}) \right), \\ \int_{A_\epsilon T} \phi = \int_{\mathbf{x}\in T} \left(\int_{\mathbf{y}\in\mathbb{R}^m} \vartheta_\epsilon(\mathbf{y}) \wedge \mathbf{S}^*_{\mathbf{y}} \phi(\mathbf{x}) \right) \end{cases} \tag{3.6}$$

[§]Symmetry was mentioned but not required in de Rham's work. But it is required in our work for the communitativity.

where ϕ is a test form. We should note that

(1) the continuity assumption about $s_{\mathbf{y}}(\mathbf{x})$ guarantees the existence of (3.6),

(2) also equations

$$\begin{cases} deg(s_{\mathbf{y}}^*(\phi)) = deg(\phi), \\ deg(\mathbf{S}_{\mathbf{y}}^*(\phi)) = deg(\phi) - 1 \end{cases} \tag{3.7}$$

imply that

$$\begin{cases} dim(R_\epsilon(T)) = dim(T), \\ dim(A_\epsilon(T)) = dim(T) - 1. \end{cases} \tag{3.8}$$

If furthermore the map

$$\begin{array}{ccc} \mathbb{R}^m \times \mathbb{R}^m & \to & \mathbb{R}^m \times \mathbb{R}^m \\ (\mathbf{x}, \mathbf{y}) & \to & (\mathbf{x}, s_{\mathbf{y}}(\mathbf{x})) \end{array}$$

is a diffeomorphism, there is a change of variables

$$\begin{cases} s_{\mathbf{y}}(\mathbf{x}) \Rightarrow \mathbf{x} \\ \mathbf{y} \Rightarrow s^{-1}(\mathbf{x}, \mathbf{y}) \end{cases} \tag{3.9}$$

(replacement of $s_{\mathbf{y}}(\mathbf{x})$ with $\mathbf{x}$; $\mathbf{y}$ with $s^{-1}(\mathbf{x}, \mathbf{y})$) where $s^{-1} : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ is C^∞ and satisfies $s_{s^{-1}(\mathbf{x},\mathbf{y})}(\mathbf{x}) = \mathbf{y}$. Then the first integral of (3.6) shows that

$$R_\epsilon T = \int_{\mathbf{x} \in T} \vartheta_\epsilon(s^{-1}(\mathbf{x}, \mathbf{y})) \tag{3.10}$$

is a C^∞ form. The form $\vartheta_\epsilon(s^{-1}(\mathbf{x}, \mathbf{y}))$ as a form in variables $\mathbf{x}, \mathbf{y}$ is the kernel (p71, [3]) of R_ϵ. We should make a note that the currents' evaluation (3.10) is defined through double currents in the same way as the fibre integrals of C^∞ forms under the projection $\mathcal{X} \times \mathcal{X} \to \mathcal{X}$.

In the step 1 we use

$$s_{\mathbf{y}}(\mathbf{x}) = \mathbf{x} + \mathbf{y}$$

for the particular case of $\mathbb{R}^m$, where the $+$ is from the standard linear structure of $\mathbb{R}^m$. Then R_ϵ is the convolution. Next we sketch the rest of two steps in the globalization, where the general $s_{\mathbf{y}}(\mathbf{x})$ will be used.

Step 2: Choose the unit ball $B \subset \mathbb{R}^m$ diffeomorphic to $\mathbb{R}^m$. Let h be the specific diffeomorphism

$$\mathbb{R}^m \quad \to \quad B,$$

defined on p66, [3]. Denote the $s_{\mathbf{y}}(\mathbf{x})$ in step 1 by $s_{\mathbf{y}}^+(\mathbf{x})$. Then we define the new C^∞ map

$$s_{\mathbf{y}}(\mathbf{x}) = \begin{cases} h s_{\mathbf{y}}^+ h^{-1}(\mathbf{x}) & \text{for} \quad \mathbf{x} \in B \\ \mathbf{x} & \text{for} \quad \mathbf{x} \notin B. \end{cases} \tag{3.11}$$

We would like to point out that $s_{\mathbf{y}}(\mathbf{x})$ satisfies assumption. Then we can define the operators $R_\epsilon^B, A_\epsilon^B$ depending on B in the same way (with a test form ϕ):

$$\begin{cases} \int_{R_\epsilon^B T} \phi = \int_{\mathbf{x}\in T} \left(\int_{\mathbf{y}\in\mathbb{R}^m} \vartheta_\epsilon(\mathbf{y}) \wedge s_{\mathbf{y}}^* \phi(\mathbf{x}) \right), \\ \int_{A_\epsilon^B T} \phi = \int_{\mathbf{x}\in T} \left(\int_{\mathbf{y}\in\mathbb{R}^m} \vartheta_\epsilon(\mathbf{y}) \wedge \mathbf{S}_{\mathbf{y}}^* \phi(\mathbf{x}) \right). \end{cases} \tag{3.12}$$

Then the operators $R_\epsilon^B, A_\epsilon^B$ will satisfy
(a) properties (1), (4), (5) and (6) in definition 3.1.
(b) $R_\epsilon^B(T)$ is C^∞ in B, $R_\epsilon^B(T) = T$ in the complement of $\bar{B}$;
(c) if T is C^r in a neighborhood of a boundary point of B, $A_\epsilon^B(T)$ will have the same regularity in the neighborhood.

Step 3: Cover the $\mathcal{X}$ with countable open sets B_i (locally finite). Now we regard each B^i as a subset of B in step 2. Let a neighborhood U_i of B_i. Let h_i be the diffeomorphic-to-image map

$$\begin{array}{ccc} U_i & \rightarrow & \mathbb{R}^m \\ \cup & & \cup \\ B_i & \rightarrow & B. \end{array}$$

Let $g_i \geq 0$ be a function on $\mathcal{X}$, which is 1 on B_i and supported in U_i. Let $T' = g_i T$ and $T'' = T - T'$. Then we let

$$\begin{aligned} R_\epsilon^i T &= (h_i^{-1})_* \circ R_\epsilon^B \circ (h_i)_* T' + T'' \\ A_\epsilon^i T &= (h_i^{-1})_* \circ A_\epsilon^B \circ (h_i)_* T'. \end{aligned}$$

(Note: h_i^{-1} is well-defined because h_i is a diffeomorphic-to-image map). Finally we glue them together by taking the composition,

$$\begin{aligned} R_\epsilon^{(N)} &= R_\epsilon^1 \circ \cdots \circ R_\epsilon^N, \\ A_\epsilon^{(N)} &= R_\epsilon^1 \circ \cdots \circ R_\epsilon^N \circ A_\epsilon^N. \end{aligned} \tag{3.13}$$

Then we take the limit as $N \to \infty$ with respect to the compact support to obtain the well-defined, global operator R_ϵ and A_ϵ.[¶]

□

Definition 3.3. *(de Rahm data)*
(a) We call R_ϵ from Theorem 3.2 the de Rham's regulator, A_ϵ from Theorem 3.2 the de Rham's homotopy operator, and the associated regularization the de Rham's regularization. All operators R_ϵ, A_ϵ are chosen to be de Rham's. (The general operators from definition 3.1 are not necessarily de Rham's).

[¶]In [3], for each open set U_i there is a different positive ϵ_i. We used the same number ϵ for all U_i. This difference should be noticed.

(b) *We define de Rham data to be all items in the construction of de Rham's regularization operators R_ϵ, A_ϵ. More specifically it includes*
(1) the covering $B_i \subset U_i$ with the order of countable i,
(2) the diffeomorphism $h_i : U_i \to \mathbb{R}^m$, and functions g_i with value 1 on B_i,
(3) for each B_i, another diffeomorphism $h^i : B^i \simeq \mathbb{R}^m$ with Euclidean coordinates,
(4) functions f_i in each $B^i \simeq \mathbb{R}^m$ as in the first step, called convolution functions, g_i, h_i called the gluing data.

(c) *The covering $B_i \subset U_i$ equipped with all the items (1)-(4) in de Rham data is called the de Rham covering. Each pair $B_i \subset U_i$ with (1)-(4) is called a de Rham chart.*

Remark The de Rham data gives a covering that regularizes the piece of T supported in B_i in each chart U_i independently and operates as an identity outside of B_i. There is "glue" (such as g_i) at each chart to glue pieces together by taking the composition. But there is no relation among pieces.

G. de Rham further showed in chapter III, §17, [3],

Corollary 3.4. *The de Rham's operator R_ϵ constructed in Theorem 3.2 is a regularizing operator, i.e. there is a C^∞ form $\varrho_\epsilon(\mathbf{x}, \mathbf{y})$ on $\mathcal{X} \times \mathcal{X}$, called the C^∞ kernel of R_ϵ, such that as currents,*

$$R_\epsilon T = \int_{\mathbf{y} \in T} \varrho_\epsilon(\mathbf{x}, \mathbf{y})$$

where the current's evaluation on the right is defined as in Theorem 9, [3] through a double form.

Remark Note: there is a sign factor when the form $\varrho_\epsilon(\mathbf{x}, \mathbf{y})$ is switched to the double form for evaluation. Kernel of an operator is a crucial technical notion defined by de Rham in [3]. We list its definition in the Appendix.

3.2 Kernel of de Rham's regulator

Definition 3.5. *Let ω be a C^∞ form of degree p on a manifold $\mathcal{X}$. We say ω is a local constant slicing, if at each point, there is an open set U containing the point such that*

$$\omega|_U = \pi^*(\theta) \tag{3.14}$$

where $\pi : U \to V \simeq \mathbb{R}^p$ is a C^∞ map, and θ is a C^∞ form on V.

Remark A form of a local constant slicing is a particular type of forms invariant under the C^∞ diffeomorphisms. For instance we notice that the differential operation commutes with the pullback π^*. Hence

$$d\omega|_U = \pi^*(d\theta) = 0$$

due to the maximal degree of θ. Therefore a form of a local constant slicing must be closed. Hence it represents a cohomology class.

Lemma 3.6. *Let $\mathcal{X}_0$ be the union of countably many proper submanifolds of dimension strictly less than dim($\mathcal{X}$). Let ω be a C^∞ form on $\mathcal{X}$ such that for each point $q \in \mathcal{X}$ there is a chart U containing q and the equality (3.14) holds on the submanifold $U - (\mathcal{X}_0 \cap U)$. Then ω is still a local constant slicing on $\mathcal{X}$.*

Proof. Let U be the neighborhood as above. By the assumption there are C^∞ forms θ of maximal degree on coordinates planes V such that

$$\omega|_{U \setminus U \cap \mathcal{X}_0} = \pi^* \bigg(\theta|_{\pi(U \setminus U \cap \mathcal{X}_0)} \bigg) \tag{3.15}$$

Notice both sides have extension to U by the continuity. Taking the closure (of topology of $\mathcal{X}$) both sides, we complete the proof. □

We'll show the kernel of R_ϵ is not only C^∞, but also a local constant slicing, therefore closed.

Proposition 3.7. *Let ϱ_ϵ be the C^∞ kernel of de Rham's regulator R_ϵ. Then ϱ_ϵ is a local constant slicing. Furthermore there is a chart U in the de Rham covering for a neighborhood of each point such that*

$$\varrho_\epsilon|_U(\mathbf{x}, \mathbf{y}) = \varrho_1|_U(\frac{\mathbf{x}}{\epsilon}, \frac{\mathbf{y}}{\epsilon}) \tag{3.16}$$

where $\mathbf{x}, \mathbf{y}$ are points in the chart.

Remark. The C^∞ kernel ϱ_ϵ is a closed form. By the homotopy formula (3.1) it represents the class of the diagonal in the cohomology group of $\mathcal{X} \times \mathcal{X}$. But ϱ_ϵ is not the de Rham's regularization of the diagonal.

Proof. For this particular local constant slicing ϱ_ϵ, we'll give a concrete description in the following. It shows that the composition (3.13) for gluing is a local fibre integral.

Denote the boundary of each local ball B_i in the de Rham data by ∂_i. Let $\partial = \sum_i \partial_i$. By Lemma 3.6, it suffices to consider the submanifold $\mathcal{X} - \partial$. So let $q \in \mathcal{X} - \partial$. Let $U_q \subset \mathcal{X} - \partial$ be a small neighborhood of q. Consider the kernel $\varrho^q_\epsilon(\mathbf{x}, \mathbf{y})$ of the de Rham's regulator

$$R_\epsilon = R^1_\epsilon \circ \cdots \circ R^n_\epsilon \tag{3.17}$$

restricted to $U_q \times U_q$, where N is finite because the covering is locally finite. Because we exclude ∂, there are two cases for the points q. If $q \notin B_i$ for some i, $R^i_\epsilon|_{U_q}$ by the definition is the identity. If $q \in B_i$ for some i, then each $R^i_\epsilon|_{U_q}$ has the C^∞ kernel $\varrho^i_\epsilon(\mathbf{x}, \mathbf{y})$ where $\mathbf{y}$ is in the second copy of U_q. Suppose there are n regulators in (3.17), and they are in the order $B_1, B_2, \cdots, B_n$. Let's denote the coordinates for each $U_i \supset B_i$ by the same letter $\mathbf{x}_i$ for which we should restrict ourselves to the domain B_i . The kernel of each R^i_ϵ is

$$\vartheta^i_1(\frac{\mathbf{x}_i}{\epsilon} \overset{i}{-} \frac{\mathbf{y}_i}{\epsilon}),$$

which means for a current T,

$$R^i_\epsilon T = \int_{\mathbf{y}_i \in T} \vartheta^i_1(\frac{\mathbf{x}_i}{\epsilon} \overset{i}{-} \frac{\mathbf{y}_i}{\epsilon})$$

where the subtraction $\overset{i}{-}$ (also $\overset{i}{+}$), scalar multiplication $\frac{\bullet}{\epsilon}$ are from the linear structure of U_i in de Rham data (they are from the de Rham data). Next we glue all pieces. The kernel ϱ_ϵ of $R_\epsilon = R^1_\epsilon \circ \cdots \circ R^n_\epsilon$ inside $B_1 \cap \cdots \cap B_n$ is the fibre integral

$$\begin{aligned}\varrho_\epsilon = \int_{(\mathbf{x}_2, \cdots, \mathbf{x}_n) \in (R^m)^{\oplus n-1}} & \vartheta^1_1(\frac{\mathbf{x}_1}{\epsilon} \overset{1}{-} \frac{\mathbf{x}_2}{\epsilon}) \wedge \vartheta^2_1(\frac{\mathbf{x}_2}{\epsilon} \overset{2}{-} \frac{\mathbf{x}_3}{\epsilon}) \wedge \cdots \\ & \wedge \vartheta^{n-1}_1(\frac{\mathbf{x}_{n-1}}{\epsilon} \overset{n-1}{-} \frac{\mathbf{x}_n}{\epsilon}) \wedge \vartheta^n_1(\frac{\mathbf{x}_n}{\epsilon} \overset{n}{-} \frac{\mathbf{y}_n}{\epsilon}),\end{aligned} \tag{3.18}$$

whose degree is m. So ϱ_ϵ is the fibre integral of the local C^∞ form,

$$\begin{array}{c}\vartheta^1_1(\frac{\mathbf{x}_1}{\epsilon} \overset{1}{-} \frac{\mathbf{x}_2}{\epsilon}) \wedge \vartheta^2_1(\frac{\mathbf{x}_2}{\epsilon} \overset{2}{-} \frac{\mathbf{x}_3}{\epsilon}) \wedge \cdots \wedge \vartheta^{n-1}_1(\frac{\mathbf{x}_{n-1}}{\epsilon} \overset{n-1}{-} \frac{\mathbf{x}_n}{\epsilon}) \wedge \vartheta^n_1(\frac{\mathbf{x}_n}{\epsilon} \overset{n}{-} \frac{\mathbf{y}_n}{\epsilon}) \\ \| \\ \vartheta^1_\epsilon(\mathbf{x}_1 \overset{1}{-} \mathbf{x}_2) \wedge \vartheta^2_\epsilon(\mathbf{x}_2 \overset{2}{-} \mathbf{x}_3) \wedge \cdots \wedge \vartheta^{n-1}_\epsilon(\mathbf{x}_{n-1} \overset{n-1}{-} \mathbf{x}_n) \wedge \vartheta^n_\epsilon(\mathbf{x}_n \overset{n}{-} \mathbf{y}_n)\end{array}$$

denoted by

$$\varsigma^{(q)}_\epsilon, \tag{3.19}$$

(of degree mn), in the projection of the Cartesian product

$$\mathcal{P}_1 : (\mathbb{R}^m)^{\oplus(n+1)} \to \mathbb{R}^m_{\mathbf{x}_1} \oplus \mathbb{R}^m_{\mathbf{y}_n} \tag{3.20}$$

where $(\mathbb{R}^m)^{\oplus(n+1)}$ have global coordinates

$$\mathbf{x}_1, \cdots, \mathbf{x}_n, \mathbf{y}_n,$$

and $\mathbb{R}^m_{\mathbf{x}_1}, \mathbb{R}^m_{\mathbf{y}_n}$ are the first and last copies. Above argument is a technical description of the kernel ϱ_ϵ.

To associated a local constant slicing form, we construct a commutative diagram by first defining the diffeomorphism

$$\begin{array}{rcc} \kappa_1 : (\mathbb{R}^m)^{\oplus n+1} & \to & (\mathbb{R}^m)^{\oplus n} \oplus \mathbb{R}^m \\ (\mathbf{x}_1, \cdots, \mathbf{x}_n, \mathbf{y}_n) & \to & (\mathbf{x}_1 \overset{1}{-} \mathbf{x}_2, \cdots, \mathbf{x}_n \overset{n}{-} \mathbf{y}_n, \mathbf{y}_n), \end{array}$$

where $\mathbf{y}_n$ are the coordinates for the last copy $\mathbb{R}^m$, and each copy $\mathbb{R}^m$ has its own linear structure. Then the projection (3.20) yields

$$\varrho_\epsilon = (\mathcal{P}_1)_*(\varsigma_\epsilon^{(q)}).$$

We denote the coordinates' components in the target space $(\mathbb{R}^m)^{\oplus n} \oplus \mathbb{R}^m$ by

$$\mathbf{x}'_1, \cdots, \mathbf{x}'_n, \mathbf{y}_n.$$

Notice the map has rank $m(n-1)$, and $\varsigma_\epsilon^{(q)}$ is the pullback form by κ_1:

$$\varsigma_\epsilon^{(q)} = \vartheta_\epsilon^1(\mathbf{x}'_1) \wedge \vartheta_\epsilon^2(\mathbf{x}'_2) \wedge \cdots \wedge \vartheta_\epsilon^n(\mathbf{x}'_n). \tag{3.21}$$

So there is a commutative diagram

$$\begin{array}{ccc} (\mathbb{R}^m)^{\oplus n+1} & \xrightarrow{\kappa_1} & (\mathbb{R}^m)^{\oplus n} \oplus \mathbb{R}^m \\ {\scriptstyle \mathcal{P}_1}\downarrow & & {\scriptstyle (\mathcal{P}_2, id)}\downarrow \\ \mathbb{R}^m_{\mathbf{x}_1} \oplus \mathbb{R}^m_{\mathbf{y}_n} & \xrightarrow{(\kappa_2, id)} & \mathbb{R}^m \oplus \mathbb{R}^m_{\mathbf{y}_n} \end{array} \tag{3.22}$$

where

$$\kappa_2 : (\mathbf{x}_1, \mathbf{y}_n) \to \mathbf{x}_1 \overset{n}{-} \mathbf{y}_n$$

and

$$\mathcal{P}_2 : (\mathbf{x}'_1, \cdots, \mathbf{x}'_n) \to \mathbf{x}'_1 \overset{1}{+} \mathbf{x}'_2 \overset{2}{+} \cdots \overset{n-1}{+} \mathbf{x}'_n$$

is the map onto the first copy $\mathbb{R}^m$. Then the commutativity of (3.22) yields

$$\varrho_\epsilon = (\mathcal{P}_1)_*(\varsigma_\epsilon^{(q)}) = (\kappa_2, id)^* \Bigg(\Big((\mathcal{P}_2, id) \circ \kappa_1 \Big)_* (\varsigma_\epsilon^{(q)}) \Bigg). \tag{3.23}$$

In (3.23), $\Bigg((\mathcal{P}_2, id) \circ \kappa_1 \Bigg)_* (\varsigma_\epsilon^{(q)})$ is a trivial pullback of a form on $\mathbb{R}^m$. Hence ϱ_ϵ is a local constant slicing. This completes the proof.

□

Example 3.8. *Let $\mathcal{X} = \mathbb{R}^n$ be equipped with the standard linear basis $\mathbf{e}_1, \cdots, \mathbf{e}_n$. Let D be the particular n dimensional coordinate's plane that transversally meets the diagonal $\Delta_{\mathbb{R}^n}$ at the origin $(\mathbf{0}, \mathbf{0})$ of $\mathbb{R}^n \times \mathbb{R}^n$. Explicitly, if $\mathbf{e}_i^1$, $\mathbf{e}_i^2$ are the standard linear bases as above, for the first and second copies of $\mathbb{R}^n$ in $\mathbb{R}^n \times \mathbb{R}^n$, then D is the subspace spanned by vectors $\mathbf{e}_i^1 - \mathbf{e}_i^2$ for $i = 1, \cdots, n$. Let*

$$\kappa : \mathbb{R}^n \times \mathbb{R}^n \to D$$

be the orthogonal projection of the product coordinates. Notice D isomorphic to $\mathbb{R}^n$ (as a subspace). So D has an isomorphic de Rham data from $\mathcal{X}$. In particular, let $d\mu$ be the Lebesgue measure of D. Let f be a C^∞ function on D with a compact support in a ball of the origin such that f is symmetric with respect to the linear structure and

$$\int_D f d\mu = 1.$$

($(\{\mathbf{e}_i\}, f)$ is a de Rham data of $\mathcal{X}$). For a positive number ϵ, the kernel ϱ_ϵ of the de Rham's regulator is

$$\kappa^* \left(\frac{1}{\epsilon^n} f(\frac{w}{\epsilon}) d\mu \right)$$

where w is the coordinate of D in the basis $\mathbf{e}_i^1 - \mathbf{e}_i^2$ for $i = 1, \cdots, n$.

4 The intersection of currents

4.1 Convergence of regularization

Theorem 4.1.

Let $\mathcal{X}$ be a manifold endowed with de Rham data. Let T_1, T_2 be two homogeneous Lebesgue currents of dimensions p, q respectively.

(1) Let ϕ be a test form of degree $p + q - m$. Then

$$\lim_{\epsilon \to 0} \int_{T_1} R_\epsilon T_2 \wedge \phi \tag{4.1}$$

exists.

(2) If ϕ is in a set of bounded forms in $\mathscr{D}(\mathcal{X})$,

$$\lim_{\epsilon \to 0} \int_{T_1} R_\epsilon T_2 \wedge \phi, \tag{4.2}$$

is bounded.

(3) Lebesgue currents are of order 0, i.e. for the Lebesgue current T and

$\phi \in \mathscr{D}(U)$, there is an estimate

$$\left|\int_T \phi\right| \leq C||\phi||_{0,K}$$

where K is a compact set of a chart U, $||\phi||_{0,K}$ is the supreme of absolute values of coefficients of ϕ in the chart, and C is a constant independent of ϕ.

Proof. (1) Let T_1, T_2 are homogeneous currents of dimensions p, q respectively. Then

$$\int_{T_1} R_\epsilon T_2 \wedge \phi = (-1)^m \int_{(T_1 \times T_2)\wedge\phi} \varrho_\epsilon(\mathbf{x}, \mathbf{y}). \tag{4.3}$$

By Proposition 3.7, the kernel $\varrho_\epsilon(\mathbf{x}, \mathbf{y})$ of R_ϵ is a local constant slicing. Thus there exists countable, locally finite open covering U of $\mathcal{X}$ such that

$$\varrho_\epsilon(\mathbf{x}, \mathbf{y})|_{U\times U} = \varrho_1(\frac{\mathbf{x}}{\epsilon}, \frac{\mathbf{y}}{\epsilon})|_{U\times U} = \pi^*(\theta(\frac{\mathbf{v}}{\epsilon})), \tag{4.4}$$

where $\pi : U \times U \to V$ is a C^∞ map to $V \simeq \mathbb{R}^m$, and θ is a C^∞ m-form on V. By a partition of unity it suffices show the convergence of (4.3) as $\epsilon \to 0$ supported in one open set U. That is the convergence of

$$\int_{(T_1\times T_2)\wedge\phi} \pi^*(\theta(\frac{\mathbf{v}}{\epsilon})). \tag{4.5}$$

where $\mathbf{v}$ is the variable of θ. Now we consider a C^∞ map

$$\pi : U \times U \to V.$$

The projection of a de Rham distribution of the current $T_1 \times T_2 \wedge \phi$ satisfies the Lebesgue condition that gives a bounded, compactly supported L^1 function $\mathcal{L}$ on V, and the Radon-Nikodym condition further implies that the limit

$$\lim_{\epsilon\to 0} \int_{\mathbf{v}\in V} \mathcal{L}(\epsilon\mathbf{v})\theta(\mathbf{v})$$

that is

$$\lim_{\epsilon\to 0} \int_{(T_1\times T_2)\wedge\phi} \pi^*(\theta(\frac{\mathbf{v}}{\epsilon})) \tag{4.6}$$

exists. We complete the proof of part (1).

(2) Now we assume ϕ is in a set in $\mathscr{D}(U)$ bounded to order 0. By the Lebesgue condition, $\mathcal{L}$ is bounded. Thus the local formula (4.6) is also bounded. Hence

$$\lim_{\epsilon\to 0} \int_{T_1} R_\epsilon T_2 \wedge \phi$$

is bounded.

(3) Continue from part (2). Notice for $T_2 = 1$, $\int_{T_1} \phi$ is equal to (4.6). Let the ϕ varies in a compact support $K \subset U$. We consider the

$$\lim_{\epsilon \to 0} \int_{(T_1 \times X) \wedge \frac{\phi}{||\phi||_{0,K}}} \pi^* \theta(\frac{\mathbf{v}}{\epsilon}).$$

Since $\frac{\phi}{||\phi||_{0,K}}$ is bounded to order 0, by the Lebesgue condition of T_1, the Lebesgue functions of $T_1 \times X \wedge \frac{\phi}{||\phi||_{0,K}}$ on V are bounded. Hence

$$\left| \lim_{\epsilon \to 0} \int_{(T_1 \times X) \wedge \frac{\phi}{||\phi||_{0,K}}} \pi^* \theta(\frac{\mathbf{v}}{\epsilon}) \right| \leq C.$$

where C is a constant independent of ϕ. Hence

$$\left| \int_{T_1} \phi \right| = \left| \lim_{\epsilon \to 0} \int_{(T_1 \times X) \wedge \phi} \pi^* \theta(\frac{\mathbf{v}}{\epsilon}) \right| \leq C ||\phi||_{0,K}.$$

We complete the proof.

□

4.2 The intersection

Definition 4.2. *Let T_1, T_2 be homogeneous Lebesgue currents on a manifold $\mathcal{X}$ endowed with de Rham data. By Theorem 4.1, the functional on $\mathscr{D}(\mathcal{X})$,*

$$\phi \to \lim_{\epsilon \to 0} \int_{T_1} R_\epsilon T_2 \wedge \phi$$

is linear, continuous. Therefore we define the intersection current

$$[T_1 \wedge T_2] \tag{4.7}$$

by the formula

$$\int_{[T_1 \wedge T_2]} \phi = \lim_{\epsilon \to 0} \int_{T_1} R_\epsilon T_2 \wedge \phi \tag{4.8}$$

for a test form ϕ. Hence there is a well-defined bilinear map, "current's intersection" satisfying

$$\begin{array}{ccc} C(X) \times C(X) & \to & \mathscr{D}'(\mathcal{X}) \\ (T_1, T_2) & \to & [T_1 \wedge T_2], \end{array}$$

dependent of de Rham data, where $\mathscr{D}'(\mathcal{X})$ denotes the space of currents and $\mathscr{L}(X)$ the subspace Lebesgue currents.

Remark The intersection $[\cdot \wedge \cdot]$ and de Rham's regularization R_ϵ, A_ϵ all depend on the de Rham data. We'll omit the notation for this dependence by fixing a data in general arguments, but will make a note in a particular case where the multiple de Rham data is necessary.

Proposition 4.3. *If T_1, T_2 are Lebesgue, so is*

$$[T_1 \wedge T_2].$$

Remark The proposition extends Proposition 2.17.

Proof. We recall and continue the setting in Theorem 4.1. By the partition of unity, we may assume

$$[T_1 \wedge T_2]$$

has a compact support in a small neighborhood U of a chart. For the Lebesgue condition we may take $\xi = 1$ and $[T_1 \wedge T_2]$ has a single de Rham distribution. Next we have a projection to set up the Lebesgue condition. Let $W \subset U$ be a coordinates plane of the dimension $dim[T_1 \wedge T_2]$, and $\pi_W : U \to W$ the projection. Then it suffices to consider the projection $(\pi_W)_*[T_1 \wedge T_2]$ which has maximal degree, so it is regarded as a distribution, denoted by $\mathcal{I}_W$. Then the functional $\mathcal{I}_W$ is

$$\phi \to \int_{(\pi_W)_*[T_1 \wedge T_2]} \phi d\mu$$

where ϕ is a test function on W and $d\mu$ is the volume form of W. According to the formula (4.6), $\mathcal{I}_W$ is equal to

$$\phi \to \lim_{\epsilon \to 0} \int_{(T_1 \times T_2) \wedge (\phi d\mu)} \pi^*(\theta(\frac{\mathbf{v}}{\epsilon})). \tag{4.9}$$

Now we rewrite the expression as follows. Recall V is the orthogonal m dimensional plane of Δ_U in $U \times U$. We project the current $T_1 \times T_2$ the plane $V \times (W \times \{0\})$ where $\{0\} \in U$ is the origin of the Euclidean space U. Notice the projection has maximal degree. Since T_1, T_2 are both Lebesgue currents, the projection regarded as a distribution is a Lebesgue function, denoted by

$$\mathcal{L}(\mathbf{v}, \mathbf{w})$$

where $\mathbf{v}, \mathbf{w}$ denote the points in V, W respectively. Then we can rewrite

$$\int_{I_W} \phi d\mu = \lim_{\epsilon \to 0} \int_{V \times (W \times \{0\})} \mathcal{L}(\epsilon \mathbf{v}, \mathbf{w}) \theta(\mathbf{v}) \phi(\mathbf{w}) d\mu. \tag{4.10}$$

Hence the distribution $\mathcal{I}_W$ satisfies the Lebesgue condition and its Lebesgue function on W, denoted by

$$\mathcal{L}_W(\mathbf{w})$$

is

$$\lim_{\epsilon\to 0}\int_{\mathbf{v}\in V}\mathcal{L}(\epsilon\mathbf{v},\mathbf{w})\theta(\mathbf{v}). \tag{4.11}$$

We should note the limit (4.11) exists due to Theorem 4.1 (part (1)). Furthermore the Radon-Nikodym condition is just the zigzag convergence of the number

$$\lim_{\epsilon\to 0}\int_{V\times(W\times\{0\})}\mathcal{L}(\epsilon\mathbf{v},D_{\boldsymbol{\lambda}}(\mathbf{w}))\theta(\mathbf{v})\phi(\mathbf{w})d\mu \tag{4.12}$$

as $\boldsymbol{\lambda} \Rsh \mathbf{0}$, where $D_{\boldsymbol{\lambda}}$ is the testing map defined in (2.3). Since $\mathcal{L}$ is an L^1 function satisfying Radon-Nikodym condition, the convergence of (4.12) indeed holds. We complete the proof.

□

Proposition 4.4. *(intersection of the supports) Let $T_1, T_2 \in C(X)$. Then*

$$supp([T_1 \wedge T_2]) \subset supp(T_1) \cap supp(T_2). \tag{4.13}$$

Proof. Suppose

$$\mathbf{a} \notin supp(T_1) \cap supp(T_2).$$

Then $\mathbf{a}$ must be outside of either $supp(T_1)$ or $supp(T_2)$. Let's assume first it is not in $supp(T_2)$. Since the support of a currents is closed, we choose a small neighborhood $U_{\mathbf{a}}$ of $\mathbf{a}$ in $\mathcal{X}$, but disjoint from $supp(T_2)$. Let ϕ be a C^∞-form of $\mathcal{X}$ with a compact support in $U_{\mathbf{a}}$. Then by Definition 3.1. when ϵ is small enough $R_\epsilon(T_2)$ is zero in $U_{\mathbf{a}}$. Hence

$$\int_{[T_1\wedge T_2]}\phi = 0, \tag{4.14}$$

for a test form ϕ supported in $U_{\mathbf{a}}$. Hence $\mathbf{a} \notin supp([T_1 \wedge T_2])$. If $\mathbf{a} \notin supp(T_1)$, $U_{\mathbf{a}}$ can be chosen disjoint with $supp(T_1)$. Then since $\phi \in \mathscr{D}(U_{\mathbf{a}})$ is a C^∞-form of $\mathcal{X}$ with a compact support in $U_{\mathbf{a}}$ disjoint with $supp(T_1)$, the restriction of ϕ to T_1 is zero. Hence

$$\int_{[T_1\wedge T_2]}\phi = 0.$$

Then $\mathbf{a} \notin supp([T_1 \wedge T_2])$. Thus

$$\mathbf{a} \notin supp(T_1) \cap supp(T_2)$$

will always imply

$$\mathbf{a} \notin supp([T_1 \wedge T_2]).$$

This completes the proof.

□

Example 4.5. *Let $\mathcal{X} = \mathbb{R}^m$ be equipped with de Rham data consisting of single open set with the convolution function f. Assume it has coordinates $x_1, \cdots, x_m$. Let*

$$T_1 = \delta_{\mathbf{0}} dx_1 \wedge \cdots \wedge dx_p, \quad 0 < p < m$$

with the δ-function $\delta_{\mathbf{0}}$ at the origin $\mathbf{0}$ of $\mathbb{R}^m$. Let T_2 be the p dimensional plane $\{x_{p+1} = \cdots = x_m = 0\}$. Now we consider the integral

$$\int_{T_1} R_\epsilon T_2. \tag{4.15}$$

By the formula (3.6), it is equal to

$$\int_{x\in T_1} \int_{y \in T_2 = \mathbb{R}^p} \frac{1}{\epsilon^m} f(\frac{x-y}{\epsilon}) dx_{p+1} \wedge \cdots \wedge dx_m \wedge dy_1 \wedge \cdots \wedge dy_p.$$

By the continuity of the functional of the currents, we can interchange the order of T_1, T_2. Thus we first evaluate T_1 at the differential form

$$\frac{1}{\epsilon^m} f(\frac{x-y}{\epsilon}) dx_{p+1} \wedge \cdots \wedge dx_m$$

to obtain that

$$\begin{array}{c} \int_{T_1} R_\epsilon T_2 \\ \| \\ (-1)^{m(m-p)} \int_{y\in\mathbb{R}^p} \frac{1}{\epsilon^m} f(\frac{-y_1}{\epsilon}, \cdots, \frac{-y_p}{\epsilon}, 0, \cdots, 0) dy_1 \wedge \cdots \wedge dy_p. \end{array} \tag{4.16}$$

Since

$$\begin{array}{l} \int_{y\in\mathbb{R}^p} \frac{1}{\epsilon^p} f(\frac{-y_1}{\epsilon}, \cdots, \frac{-y_p}{\epsilon}, 0, \cdots, 0) dy_1 \wedge \cdots \wedge dy_p \\ = (-1)^p \int_{y\in\mathbb{R}^p} f(y_1, \cdots, y_p, 0, \cdots, 0) dy_1 \wedge \cdots \wedge dy_p \end{array} \tag{4.17}$$

is a non-zero constant, $\int_{T_1} R_\epsilon T_2$ diverges to infinity as $\epsilon \to 0$. Hence the intersection $[\cdot \wedge \cdot]$ does not exist for such T_1, T_2.

Example 4.6. *(Deligne) Let $\mathcal{X} = \mathbb{R}^2$ be equipped with the de Rham data that has a single chart $\mathbb{R}^2$ with the convolution function f. Let A be the current of the upper half plane, B the current of the lower half plane, and $\delta_{\mathbf{0}}$ the current of delta function at $\{\mathbf{0}\}$. Let $b = \int_B f d\mu$ and $a = \int_A f d\mu$ where $d\mu$ is the Euclidean measure for the plane. Notice a, b could be any real number dependent of de Rham data. Then*

$$[B \wedge \delta_{\mathbf{0}}] = b\delta_{\mathbf{0}} \text{ (by the direct computation)} \tag{4.18}$$

$$\Big[A \wedge [B \wedge \delta_{\mathbf{0}}]\Big] = ab\delta_{\mathbf{0}} \ (follows\ from\ (4.18)) \tag{4.19}$$

$$[A \wedge B] = 0 \text{ (since it is supported on a lower dimension)} \tag{4.20}$$

$$\Big[[A \wedge B] \wedge \delta_{\mathbf{0}}\Big] = 0 \ (follows\ from\ (4.20)) \tag{4.21}$$

So

$$\Big[A \wedge [B \wedge \delta_{\mathbf{0}}]\Big] \neq \Big[[A \wedge B] \wedge \delta_{\mathbf{0}}\Big].$$

Hence the intersection $[\cdot \wedge \cdot]$ *is not associative.*

Remark It is also expected that the intersection is not commutative.

A Appendix: Kernel

In [3] de Rham created the notion of "regularizing operator" which includes de Rham's regulator R_ϵ. Let $\mathcal{X}, \mathcal{Y}$ be two manifolds. Let $L \in \mathscr{D}'(\mathcal{X} \times \mathcal{Y})$. There is a homomorphism

$$\begin{array}{ccc} \mathscr{D}(\mathcal{X}) \times \mathscr{D}(\mathcal{Y}) & \rightarrow & \mathbb{R} \\ (\phi_x, \phi_y) & \rightarrow & \int_L \phi_x \wedge \phi_y. \end{array} \tag{A.1}$$

It leads to another homomorphism

$$\Lambda ; \mathscr{D}(\mathcal{X}) \quad \rightarrow \quad \mathscr{D}'(\mathcal{Y}) \tag{A.2}$$

Then L is called the kernel of Λ. Conversely given a homomorphism Λ, there is a kernel current L on $\mathcal{X} \times \mathcal{Y}$. Notice

$$\begin{array}{cc} \mathscr{D}(\mathcal{X}), & \mathscr{E}(\mathcal{Y}) \\ \cap & \cap \\ \mathscr{E}'(\mathcal{X}), & \mathscr{D}'(\mathcal{Y}) \end{array} \tag{A.3}$$

where $\mathscr{E}(\bullet)$ is the set of C^∞ forms, and $'$ is the topological dual.

Definition A.1. *(1) If* Λ *can be extended to a continuous homomorphism*

$$\Lambda : \mathscr{E}'(\mathcal{X}) \quad \rightarrow \quad \mathscr{D}'(\mathcal{Y}) \tag{A.4}$$

we say Λ *is regular.*

(2) If furthermore, the regular Λ *has the image inside of* $\mathscr{E}(\mathcal{Y})$*, i.e.*

$$\Lambda : \mathscr{E}'(\mathcal{X}) \quad \rightarrow \quad \mathscr{E}(\mathcal{Y}) \tag{A.5}$$

we say Λ *is regularizing.*

Theorem A.2. *(de Rham)*
Λ *is regularizing if and only if the kernel* L *is a* C^∞ *form on* $\mathcal{X} \times \mathcal{Y}$*. In particular* R_ϵ *is regularizing.*

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