Rational curves on a generic Calabi-Yau complete intersection of dimension 3

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Abstract

We prove the following results. If X_3 is a generic Calabi-Yau complete intersection of dimension 3,

(1) then for each natural number d there exists a rational map $c \in Hom_{bir}(\mathbf{P}^1, X_3)$ of $deg(c(\mathbf{P}^1)) = d$,

(2) further more all such c are immersions satisfying

$$N_{c(\mathbf{P}^1)/X_3} \simeq \mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1). \tag{0.1}$$

Contents

1 Statement

In mirror symmetry there is a general consensus that a "generic" Calabi-Yau 3-fold over $\mathbb{C}$ should contain and only contain finitely many irreducible rational curves of each degree with respect to the polarization. In this paper let's consider the case of complete intersections.

Theorem 1.1.
Let X_3 be a generic, Calabi-Yau complete intersection of dimension 3 over $\mathbb{C}$.
Then
(1) X_3 admits an irreducible rational curve C of each degree,
(2) all such $C \subset X_3$ are immersions and the normal bundle N_{C/X_3} is isomorphic to

$$\mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1). \tag{1.1}$$

2 Sketch of the proof

2.1 Setting

Throughout the paper rational curves are curves rationally parametrized by $\mathbf{P}^1$. The image is called an irreducible rational curve.

Rational curves on projective varieties has been a topic for many decades. The general theory which has its own technique is not the focus of this paper. Instead we are interested in a specific type of problems with a quite different technique.

Focus 2.1. *Which generic complete intersection X admits an irreducible rational rational curve C of each degree?*

Focus 2.2. *Once the first question is affirmative, what is the normal sheaf*

$$N_{C/X}? \tag{2.1}$$

Theorem 1.1 tries to answer these two questions in the case of Calabi-Yau 3-folds.

The idea of the work starts from and stays in a down-to-earth setting, which employees linear algebra only. The method first converts the invariant expression of Theorem 1.1 to a variant expression as the content of Theorem 1.1 stays the same. Then it explores the unique linear algebra in the variant setting to reach an algebraic result. At last it converts the algebraic result back to the invariants. In technique the first conversion

$$\text{I}nvariant \Rightarrow Variant \tag{2.2}$$

uses classical geometry. The second conversion

$$\text{V}ariant \Rightarrow invariant \tag{2.3}$$

uses Clemens' deformation idea [1].

Let's start with this alternative setting. Let

$$M_d = (H^0(\mathcal{O}_{\mathbf{P}^1}(d))^{\oplus(n+1)} \simeq \mathbb{C}^{(n+1)(d+1)}. \tag{2.4}$$

The open set $M_{bir,d}$ of M_d represents (but is not equal to)

$$\{c \in Hom_{bir}(\mathbf{P}^1, \mathbf{P}^n) : deg(c(\mathbf{P}^1)) = d\}.$$

Let

$$g_i, i = 1, \cdots, r = n-3$$

be sections in $H^0(\mathcal{O}_{\mathbf{P}^n}(h_i))$ and

$$\sum_{i=1}^{r} h_i = n+1. \tag{2.5}$$

Let

$$X_i = \cap_{k=1}^{k=n-i} div(g_k). \tag{2.6}$$

In this paper, the Cartesian product

$$(g_1, \cdots, g_{n-i})$$

is also called a complete intersection of type $(h_1, \cdots, h_{n-i})$. So

$$X_3 = \cap_{i=1}^{r} div(g_i) \tag{2.7}$$

is a complete intersection Calabi-Yau 3-fold in the usual sense for generic

$$(g_1, \cdots, g_r) \in \prod_{i+1}^{r} H^0(\mathcal{O}_{\mathbf{P}^n}(h_i)).$$

Choose distinct $h_i d + 1$ points $t_j^i \in \mathbf{P}^1$. Let

$$\mathbf{t}^i = (t_1^i, \cdots, t_{h_i d+1}^i) \in Sym^{h_i d+1}(\mathbf{P}^1). \tag{2.8}$$

In the rest of the paper, we'll use following conventions in affine coordinates.

(a) t_j^i or t denotes a complex number which is a point in an affine open set $\mathbb{C} \subset \mathbf{P}^1$,

(b) $c(t)$ denotes the image

$$\begin{array}{ccc} \mathbb{C} & \stackrel{c}{\to} & \mathbb{C}^{(n+1)(d+1)} \\ t & \to & c(t), \end{array}$$

(c) $g_i \in H^0(\mathcal{O}_{\mathbf{P}^n}(h_i))$ is a homogeneous polynomial of degree h_i in $n+1$ variables.

We should note that in these affine coordinates, the incidence relation $c^*(g_i) = 0$ can be expressed as the composition $g_i(c(t)) = 0$ for all

$$t \in \mathbb{C}.$$

Let C_M be a system of affine coordinates for M_d, which determines an isomorphism

$$M_d \simeq \mathbb{C}^{(n+1)(d+1)}.$$

We define a system of polynomials in the variable $c \in M_d$

$$g_i(c(t_j^i)), i = 1, \cdots, r, j = 1, \cdots, h_i d + 1. \tag{2.9}$$

Then the subsets of polynomials

$$g_1, \cdots, g_l, l \leq r$$

give a rise to a holomorphic map μ_l

$$\mu_l : M_d \simeq \mathbb{C}^{(n+1)(d+1)} \to \mathbb{C}^{m_l d + l}, \tag{2.10}$$

where $m_l = \sum_{i=1}^{l} h_i$. We"ll denote

$$I_l = \mu_l^{-1}(0) \tag{2.11}$$

which will be called the incidence scheme of rational curves on the complete intersection of $g_1, \cdots, g_l$. Then the differential map $(\mu_l)_*$ is represented by the Jacobian matrix of size

$$\left(m_l d + l\right) \times \left(n+1\right)\left(d+1\right)$$

denoted by J_l, which depends on $g_i, \mathbf{t}^j$. However once the points $\mathbf{t}^j$ are fixed, the matrix J_l is well-defined and varied algebraically on the entire affine space $M_d \times \mathbb{A}$, where $\mathbb{A} \subset \prod_i \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h_i)))$ is affine. We call it the Jacobian matrix of the incidence scheme I_l.

The scheme I_l, which could be reducible with multiple dimensions,[1] is the alternative to various moduli spaces of rational curves and maps. In this paper we show a methodology in the calculation of the Jacobian matrix J_l at a point of I_l corresponding to an irreducible rational curve on generic complete intersections. The components containing such points always have the smallest dimension as expected. They correspond to the components that have actual fundamental classes instead of virtual fundamental classes in Mirror symmetry. The methodology is rooted in a specific pattern of the Jacobian matrix J_l modeled on the Vandermonde matrices.

In general we are interested in the following questions,

(1) what is the dimension of I_l?,

(2) is it reduced?

(3) is it irreducible, and if not, what is the structure of each component?

Answers to these questions would solve some technical problems in Mirror symmetry, that have been left out of the physics' formulation. Thus it is an

[1]For instance, it contains the components of multiple covering maps of $\mathbf{P}^1$.

alternative to patch some holes in the general mathematical theory. The complete answers to these questions are out of scope of this paper. Here we are going to use linear algebra to explore the dimension of I_l for some complete intersections.

2.2 Existence

In the first part, we prove the existence of irreducible rational curves C on a generic complete intersection Calabi-Yau 3-fold X_3. We avoid a direct construction.

(I) First we'll use the proven existence of irreducible rational curves of arbitrary degrees on a single generic hypersurface of lower degrees (which are Fano). After its extension to a special complete intersection in $\mathbf{P}^n$ by joining more Fano hypersurfaces, we use linear algebra to glue the block matrix for each Fano hypersurface to obtain the non-degeneracy of Jacobian matrix J_r. This will show the existence of an irreducible rational curve C' of each degree on a special complete intersection Calabi-Yau 3-fold X'_3 with the non-degenerate Jacobian J'_r at the point (C', X'_3). [2]

(II) Applying Clemens' deformation idea, this smooth C' is deformed to generic X_3 as a different irreducible rational curve $C \subset X_3$. The requirement for such a deformation is the non-degeneracy of Jacobian matrix J_l.

2.3 Rigidity

The second step is to show the normal bundle of all such irreducible curves $C \subset X_3$ are split as in (0.1). This rigidity is determined by the dimension of the incidence scheme I_r. First we have the general study of a uniruled projective variety to deduce that a free rational curve on it has the unobstructed deformation on its generic hypersurfaces. This no-trivial result follows from [7] or [8]. Then we notice that the filtration of complete intersections

$$X_3 \subset \cdots \subset X_0 \simeq \mathbf{P}^n \tag{2.12}$$

[2]For curves of a fixed and small degree, computer software has been used to find its Jacobian. See p. 295, [2].

corresponds to another filtration of subvarieties

$$I_r|_c \subset I_{r-1}|_c \subset \cdots \subset I_1|_c \subset I_0|_c, \tag{2.13}$$

where the subscript $|_c$ means the analytic neighborhood around the point c with $c(\mathbf{P}^1) = C$. Then we notice all X_i for $i \neq 3$ are Fano, therefore uniruled. Applying above deformation result for rational curve on uniruled varieties, we obtain a recursive formula

$$dim(I_l|_c) = dim(I_{l+1}|_c) - (h_l d + 1). \tag{2.14}$$

(or equivalently $dim(H^0(c^*(T_{X_l})) = dim(H^0(c^*(T_{X_{l+1}})) - (h_l d + 1).)$. Applying the Calabi-Yau condition (2.5) we obtain that

$$dim(I_r|_c) = 4. \tag{2.15}$$

(or $dim(H^0(c^*(T_{X_3})) = 3)$.

Then the dimension $dim(H^0(c^*(T_{X_3})) = 3$ forces c to be an immersion and furthermore to be rigid.

We organize the rest of the paper as follows. In section 3, we prove the existence, and in section 4 we prove the rigidity. Appendix covers a particular technique in linear algebra for the existence.

3 Existence of rational curves

Starting from this section we give technical proofs. This section focuses on the existence.

We resume all notations introduced in section 1. Recall $g_i, i = 1, \cdots, r$ are sections in $H^0(\mathcal{O}_{\mathbf{P}^n}(h_i))$ such that $n - 3 = r$ and $n + 1 = \sum_i h_i$. Let

$$X_l = \cap_{i=1}^{n-l} div(g_i), l \leq r \tag{3.1}$$

In particular, if

$$(g_1, \cdots, g_r) \in \prod_{i=1}^{r} H^0(\mathcal{O}_{\mathbf{P}^n}(h_i))$$

is generic, X_3 is a smooth complete intersection Calabi-Yau 3-fold.

In this subsection, we prove part (1) of Theorem 1.1. It asserts the existence of an irreducible rational curve of each degree d on the generic X_3.

Theorem 3.1. *There exist irreducible rational curves of each degree d on the following generic complete intersection Calabi-Yau 3-folds.*

$$(2,2,2,2) \ \textit{type in } \mathbf{P}^7 \tag{3.2}$$

$$(3,2,2) \ \textit{type in } \mathbf{P}^6, \tag{3.3}$$

$$(3,3) \ \textit{and } (4,2) \ \textit{types in } \mathbf{P}^5 \tag{3.4}$$

$$(5) \ \textit{type}, \textit{i.e.quintic } 3\text{-}fold \ \textit{in } \mathbf{P}^4. \tag{3.5}$$

The case (3.5) has been proved by H. Clemens [1] and S. Katz ([3]). Let's consider other 3 cases. Suppose that in all 3 cases, there exist a special complete intersection, denoted by sections $g_1', \cdots, g_r'$ and a smooth rational curve c_g of degree d with $c_g^*(g_i') = 0$ for all i such that the corresponding Jacobian matrix J_r at c_g has full rank. Then we divide the coordinates of M_d to two parts, c_{ind} and c_{free}. The partial derivatives with respect to c_{ind} form the maximal minor block J_{ind} in the Jacobian matrix J_r. Thus J_{ind} is a maximal non-degenerate block. The size of J_{ind} is therefore

$$\Big((n+1)d + r\Big) \times \Big((n+1)d + r\Big)$$

Let the remaining coordinates of M_d be c_{free}, and corresponding block matrix in J_r is J_{free}. Hence $(c_{ind}c_{free})$ are the affine coordinates of M_d and $(J_{ind}J_{free}) = J_r$. Since J_{ind} evaluated at the special complete intersection (g_i') and smooth c_g is non-degenerate, by the implicit function theorem in complex analysis, there exists analytic functions near c_g, $c_{ind} = \alpha_{ind}(c_{free}, g_1, \cdots, g_r)$ (g_i are locally free sections) such that

$$g_i(c(t_j^i)) = 0, \ \textit{for all } i, j \tag{3.6}$$

where $c = \alpha_{ind}(c_{free}, g_1, \cdots, g_r)c_{free}$. In geometric term, this means that there exists a smooth rational curve c of degree d on each same type of complete intersection (free choice of g_i) near the special one (g_i'). Since these deformed complete intersections (g_i) are all generic in Zariski-topology, we complete the proof of the existence in these cases. So in the following subsections we are going to find such a special complete intersection in each case.

3.1 Complete intersection of $(2,2,2,2)$ type in $\mathbf{P}^7$

Proposition 3.2. *There is a smooth rational curve C_g of each degree d on a special complete intersection X_3 of type $(2,2,2,2)$ in $\mathbf{P}^7$ such that $J_4|_{c_g}$ at (c_g, X_3) is non-degenerate, where c_g is the normalization of C_g.*

Proof. Let $[z_0, \cdots, z_7]$ be homogeneous coordinates of $\mathbf{P}^7$. Let $\mathbf{P}^3$ the subspace defined by $z_4 = z_5 = z_6 = z_7 = 0$. First we consider a generic quadric $g \in H^0(\mathcal{O}_{\mathbf{P}^3}(2))$ in $\mathbf{P}^3$. By [4], Hilbert scheme $Hilb_{c(\mathbf{P}^1)}(div(g))$ at a smooth rational curve

$$c \in Hom_{bir}(\mathbf{P}^1, \mathbf{P}^n)$$

of degree d is non-empty and smooth with the expected dimension. Then I_1 is smooth at c with expected dimension. Let J_1' be the corresponding Jacobian matrix of I_1 in $\mathbf{P}^3$ (as defined in the introduction). By Lemma A.2, it has full rank. In the following we use the "gluing" technique for block matrices to extend the Jacobian matrix J_1' to $\mathbf{P}^7$. Assume $c = [c_0, c_1, c_2, c_3]$. It is extended to a smooth rational curve c_g in $\mathbf{P}^7$ as follows

$$c_g = [c_0, c_1, c_2, c_3, c_0, c_1, c_2, c_3], \tag{3.7}$$

which is isomorphic to c. We define the special complete intersection of $(2,2,2,2)$ type as follows. Let

$$\begin{aligned} g_1 &= g(\textit{by the extension}), \\ g_2 &= g(z_4, z_5, z_2, z_3) \\ g_3 &= g(z_0, z_1, z_6, z_7), \\ g_4 &= z_6^2 + z_7^2 - z_2^2 - z_3^2. \end{aligned} \tag{3.8}$$

be four quadrics in $\mathbf{P}^7$. Then we have $c_g^*(g_i) = 0$ for all i.

Let $\theta_1, \theta_3, \theta_5, \theta_7$ be the affine coordinates for $2nd, 4th, 6th$ and $8th$ copies $H^0(\mathcal{O}_{\mathbf{P}^1}(d))$ in M_d. Let V_0 be the hyperplane of $H^0(\mathcal{O}_{\mathbf{P}^1}(d))$, whose elements vanish at 0. Let $\theta_0, \theta_2, \theta_4, \theta_6$ be the affine coordinates for V_0 in the $1st, 3rd, 5th, 7th$ components of M_d. The we'll show these variables, that are not all variables of M_d, are c_{ind}. This amounts to show that the Jacobian matrix has full rank. Thus we first write down the blocks of Jacobian matrix.

We let

$$\begin{aligned} B_{11} &= \frac{\partial(g_1(c_g(t_1^1)),\cdots,g_1(c_g(t_{2d+1}^1)))}{\partial(\theta_0,\theta_1)} \\ B_{12} &= \frac{\partial(g_1(c_g(t_1^1)),\cdots,g_1(c_g(t_{2d+1}^1)))}{\partial(\theta_2,\theta_3)} \\ B_{22} &= \frac{\partial(g_2(c_g(t_1^2)),\cdots,g_1(c_g(t_{2d+1}^2)))}{\partial(\theta_2,\theta_3)} \\ B_{23} &= \frac{\partial(g_2(c_g(t_1^2)),\cdots,g_2(c_g(t_{2d+1}^2)))}{\partial(\theta_4,\theta_5)} \\ B_{33} &= \frac{\partial(g_3(c_g(t_1^3)),\cdots,g_3(c_g(t_{2d+1}^3)))}{\partial(\theta_4,\theta_6)} \\ B_{34} &= \frac{\partial(g_3(c_g(t_1^3)),\cdots,g_3(c_g(t_{2d+1}^3)))}{\partial(\theta_6,\theta_7)} \\ B_{42} &= \frac{\partial(g_4(c_g(t_1^4)),\cdots,g_g(c_g(t_{2d+1}^4)))}{\partial(\theta_2,\theta_3)} \\ B_{44} &= \frac{\partial(g_4(c_g(t_1^4)),\cdots,g_4(c_g(t_{2d+1}^4)))}{\partial(\theta_6,\theta_7)} \end{aligned} \tag{3.9}$$

Then one of maximal minor blocks J_{ind} of the Jacobian matrix J_4 (in $\mathbf{P}^7$) at c_g is formed by the block matrices

$$J_{ind} = \begin{pmatrix} B_{11} & B_{12} & 0 & 0 \\ 0 & B_{22} & B_{23} & 0 \\ 0 & 0 & B_{33} & B_{34} \\ 0 & B_{42} & 0 & B_{44} \end{pmatrix} \tag{3.10}$$

We can verify that B_{ij} in (3.10) satisfy all conditions in Lemma A.2. Furthermore all conditions in Lemma A.6 for $J|_{c_g}$ are also satisfied. By the Lemmas $J|_{c_g}$ is non-degenerate. We complete the proof. □

3.2 Complete intersection of $(3, 2, 2)$ type in $\mathbf{P}^6$

Proposition 3.3. *There is a smooth rational curve C_g of each degree d on a special complete intersection X_3 of type $(3, 2, 2)$ in $\mathbf{P}^6$ such that $J_3|_{c_g}$ at (c_g, X_3) is non-degenerate, where c_g is the normalization of C_g.*

Proof. Let $z_0, \cdots, z_6$ be homogeneous coordinates of $\mathbf{P}^6$. Let $\mathbf{P}^3 \subset \mathbf{P}^6$ be the subspace defined by $z_4 = z_5 = z_6 = 0$. By [4], there exists a generic

$$g \in H^0(\mathcal{O}_{\mathbf{P}^3}(2))$$

such that it contains a smooth rational curve

$$c = [c_0, c_1, c_2, c_3]. \tag{3.11}$$

We define a new smooth rational curve of degree d to be

$$c_g = [c_0, c_1, c_2, c_3, c_0, c_1, c_2]. \tag{3.12}$$

We define hypersurfaces as

$$\begin{aligned} g_1 &= z_0 g(z_0, z_1, z_2, z_3) + (z_0 - z_4)\alpha(z_0, z_1, z_2, z_3) \\ g_2 &= g(z_4, z_5, z_2, z_3) \\ g_2 &= g(z_4, z_5, z_6, z_3) \end{aligned} \tag{3.13}$$

where α is a generic quadric with respect to g. Let $\theta_1, \cdots, \theta_7$ be the affine coordinates for 7 copies $H^0(\mathcal{O}_{\mathbf{P}^1}(d))$ in M_d. Let

$$\begin{aligned} \mathbf{t}^1 &= (t_1^1, \cdots, t_{3d+1}^1) \in sym^{3d+1}(\mathbf{P}^1) \\ \mathbf{t}^2 &= (t_1^2, \cdots, t_{2d+1}^2) \in sym^{2d+1}(\mathbf{P}^1) \\ \mathbf{t}^3 &= (t_1^3, \cdots, t_{2d+1}^3) \in sym^{2d+1}(\mathbf{P}^1). \end{aligned} \tag{3.14}$$

We write down the block matrices for the Jacobian matrix. Let

$$\begin{aligned} B_{1j} &= \frac{\partial(g_1(c_g(t_1^1)), \cdots, g_1(c_g(t_{3d+1}^1)))}{\partial\theta_j} \\ B_{2j} &= \frac{\partial(g_2(c_g(t_1^2)), \cdots, g_2(c_g(t_{2d+1}^2)))}{\partial\theta_j} \\ B_{3j} &= \frac{\partial(g_3(c_g(t_1^3)), \cdots, g_3(c_g(t_{2d+1}^3)))}{\partial\theta_j}. \end{aligned} \tag{3.15}$$

Next we write down the Jacobian matrix directly as

$$J_3 = \begin{pmatrix} B_{11} & B_{12} & B_{13} & B_{14} & B_{15} & 0 & 0 \\ 0 & 0 & B_{23} & B_{24} & B_{25} & B_{26} & 0 \\ 0 & 0 & 0 & B_{34} & B_{35} & B_{36} & B_{37} \end{pmatrix} \tag{3.16}$$

Then we verify B_{ij} satisfy all conditions in Lemmas A.2, A.3. Then we can apply Lemma A.5. We obtain that Jacobian matrix J_3 of the incidence scheme has full rank. We complete the proof of this case. $\square$

3.3 Complete intersections of $(3,3)$ and $(4,2)$ types in $\mathbf{P}^5$

Proposition 3.4. *There is a smooth rational curve C_g of each degree d on a special complete intersection X_3 of type $(3,3)$ in $\mathbf{P}^5$ such that $J_2|_{c_g}$ at (c_g, X_3) is non-degenerate, where c_g is the normalization of C_g.*

Proof. Let $z_0, \cdots, z_5$ be homogeneous coordinates of $\mathbf{P}^5$. Let $\mathbf{P}^3 \subset \mathbf{P}^5$ be the subspace defined by $z_4 = z_5 = 0$. By [4], there exists a generic $g \in H^0(\mathcal{O}_{\mathbf{P}^3}(2))$ such that it contains an irreducible rational curve

$$c = [c_0, c_1, c_2, c_3] \tag{3.17}$$

of degree d. Next we define a smooth rational curve of degree d in $\mathbf{P}^5$ as

$$c_g = [c_0, c_1, c_2, c_3, c_1, c_0]. \tag{3.18}$$

We also define two cubic hypersurfaces

$$\begin{aligned} g_1 &= z_0 g(z_0, z_1, z_2, z_3) + (z_0 - z_5)\alpha(z_0, z_1, z_2, z_3) \\ g_2 &= z_0 g(z_5, z_4, z_2, z_3) + (z_1 - z_4)\alpha(z_0, z_1, z_2, z_3), \end{aligned} \tag{3.19}$$

where α is a generic quadric with respect to g. Let $\theta_1, \cdots, \theta_6$ be the affine coordinates for 6 copies $H^0(\mathcal{O}_{\mathbf{P}^1}(d))$ in M_d (all variables of M_d). Let

$$\begin{aligned} \mathbf{t}^1 &= (t^1_1, \cdots, t^1_{3d+1}) \in sym^{3d+1}(\mathbf{P}^1) \\ \mathbf{t}^2 &= (t^2_1, \cdots, t^2_{3d+1}) \in sym^{3d+1}(\mathbf{P}^1). \end{aligned} \tag{3.20}$$

Let

$$B_{ij} = \frac{\partial(g_i(c_g(t^i_1)), \cdots, g_i(c_g(t^i_{3d+1})))}{\partial \theta_j} \tag{3.21}$$

for $i = 1, 2$, $j = 1, \cdots, 6$. Then we directly write down the Jacobian J_2 of the incidence scheme at c_g.

$$J_2 = \begin{pmatrix} B_{11} & B_{12} & B_{13} & B_{14} & 0 & B_{16} \\ 0 & B_{22} & B_{23} & B_{24} & B_{25} & B_{26} \end{pmatrix} \tag{3.22}$$

Similarly conditions of Lemma A.3 are met. By Lemma A.4, it has full rank. We complete the proof in this case. □

Proposition 3.5. *There is a smooth rational curve C_g of each degree d on a special complete intersection X_3 of type $(4,2)$ in $\mathbf{P}^5$ such that $J_2|_{c_g}$ at (c_g, X_3) is non-degenerate, where c_g is the normalization of C_g.*

Proof. As in the case 1, $z_0, \cdots, z_5$ are homogeneous coordinates of $\mathbf{P}^5$. Let $\mathbf{P}^4$ be the subspace covered by the coordinates

$$[z_0, z_1, z_2, z_3, z_4, 0].$$

Let $\mathbf{P}^3$ be the subspace covered by coordinates

$$[z_0, z_1, z_2, z_3, 0, 0].$$

By the Mori's result [6]. there is a smooth rational curve c_s of degree d on a generic quartic $g \in H^0(\mathcal{O}_{\mathbf{P}^3}(4))$. Let

$$\mathbf{t}^1 = (t_1^1, \cdots, t_{4d}^1) \in sym^{4d}(\mathbf{P}^1) \tag{3.23}$$

be generic.

Then the Jacobian matrix J'

$$J' = \frac{\partial\big(g(c_s(t_1^1)), \cdots, g(c_s(t_{4d}^1))\big)}{\partial(\theta_0, \cdots, \theta_3)} \tag{3.24}$$

has full rank. [3] Let's extend it to $\mathbf{P}^4$. Define a new quartic by setting $g_1 = g + z_0^3 z_4$ in $\mathbf{P}^4$ and new rational curve by setting

$$c = [c_0, c_1, c_2, c_3, 0]$$

where $[c_0, c_1, c_2, c_3]$ is the Mori's curve. Next we add one generic point

$$t_{4d+1}^1 \in \mathbf{P}^1,$$

and extend the coordinates of

$$H^0(\mathcal{O}_{\mathbf{P}^1}(d))^{\oplus 4}$$

[3]The number of points t_j^1 is unusual. This exceptional case holds only for quartic surface in $\mathbf{P}^3$.

to

$$H^0(\mathcal{O}_{\mathbf{P}^1}(d))^{\oplus 5}.$$

These add the a new row –the differential of $\mathbf{d}g_1(c(t^1_{4d+1}))$ to the Jacobian J', Hence we obtain a new Jacobian matrix J_1 in $\mathbf{P}^4$ at c,

$$J_1|_{c_g} = \begin{pmatrix} J' & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \tag{3.25}$$

where

$$\begin{pmatrix} B_{21} & B_{22} \end{pmatrix} \tag{3.26}$$

is the differential 1-form $\mathbf{d}g_1(c(t^1_{4d+1}))$. Hence $J_1|_c$ in $\mathbf{P}^4$ has full rank. To summarize it, we found a special quartic g_1 in $\mathbf{P}^4$ containing a smooth rational curve c of the given degree and its Jacobian matrix is of full rank. [4]

Next we extend it one more time to $\mathbf{P}^5$. Let g_1 be the extension of original g_1 to $\mathbf{P}^5$. Let $g_2 = z_0z_4 + z_1z_5$ and the

$$c_g = [c_0, c_1, c_2, c_3, 0, 0]$$

be the smooth curve in $\mathbf{P}^5$. Thus c_g lies on the complete intersection of g_1, g_2. We add new set of generic $2d+1$ points in $\mathbf{P}^1$,

$$\mathbf{t}^2 = (t^2_1, \cdots, t^{2d+1}_1) \in sym^{2d+1}(\mathbf{P}^1). \tag{3.27}$$

Then we can write down the Jacobian matrix J_2 in this case. It is equal to

$$J_2 = \begin{pmatrix} J_1 & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{3.28}$$

where

$$\begin{pmatrix} A_{21} & A_{22} \end{pmatrix} \tag{3.29}$$

is the Jacobian matrix at c_g of $2d+1$ many functions (in c),

$$g_2(c(t^2_1)), \cdots, g_2(c(t^2_{2d+1})) \tag{3.30}$$

and A_{22} is the block with respect to $2d+2$ coordinates in last two components of

$$H^0(\mathcal{O}_{\mathbf{P}^1}(d))^{\oplus 6}. \tag{3.31}$$

By Lemma A.2, A_{22} has full rank. Hence the Jacobian matrix J_2 also has full rank. We complete the proof. □

[4]We added one more feature to Mori's existence. That is the non-degeneracy of the Jacobian matrix.

3.4 Quintic in $\mathbf{P}^4$

This has been proved in [3]. We'll repeat the same construction in [1], but continue with our method. By the Mori's result [6]. there is a smooth rational curve c_s of degree d on a generic quartic $div(g)$ in $\mathbf{P}^3$. Now we extend it to $\mathbf{P}^4$ by setting

$$c_g = [c_0, c_1, c_2, c_3, 0], for\ c_s = [c_0, c_1, c_2, c_3], \tag{3.32}$$

and the new quintic

$$g_1 = lg + qz_4, \tag{3.33}$$

where l is a generic linear polynomial and q is a generic quartic. By [9], the corresponding Jacobian matrix J_1 has full rank.

Corollary 3.6. *Let X_3 be a generic complete intersection Calabi-Yau 3-fold. Then it contains an irreducible rational curve of each degree d.*

Proof. X_3 is projectively embedded as one of the complete intersections in Theorem 3.1. Therefore the corollary follows.

□

Remark In above arguments for the existence, all rational curves are smooth. But non-smooth and irreducible rational curves on a generic complete intersection Calabi-Yau 3-fold do exist.

4 Rigidity of rational curves

4.1 Rational curves with unobstructed deformation

In this subsection we let Y be an arbitrary smooth projective variety over $\mathbb{C}$. We'll prove a general assertion in Theorem 4.3 about uniruled variety. For a parametrized rational curves on Y, the notion of having unobstructed deformation is weaker than being a free morphism. Precisely let $c \in Hom_{bir}(\mathbf{P}^1, Y)$.

Definition 4.1. *If the normal sheaf $c^*(N_{c(\mathbf{P}^1)/Y})$ denoted simply by $N_{c/Y}$ has the vanishing first cohomology, i.e.*

$$H^1(N_{c/Y}) = 0, \tag{4.1}$$

then we say c has unobstructed deformation on Y.

Since $H^1(N_{c/Y}) = 0$ is equivalent to $H^1(c^*(T_Y)) = 0$, then Definition 4.1 is equivalent to the following splitting

$$c^*(T_Y) \simeq \oplus_j \mathcal{O}_{\mathbf{P}^1}(a_j), a_j \geq -1. \tag{4.2}$$

On the other hand, we define

Definition 4.2. *If $c^*(T_Y)$ is generated by global sections, we say c is a free morphism, and Y is uniruled.*

This is a special case of the more general definition in [5].

It is clear that c being free is equivalent to the splitting

$$c^*(T_Y) \simeq \oplus_j \mathcal{O}_{\mathbf{P}^1}(a_j), a_j \geq 0. \tag{4.3}$$

So being free is stronger than having unobstructed deformation.

Theorem 4.3. *Let $\mathcal{L} \simeq \mathcal{O}_{\mathbf{P}^n}(1)|_Y$ be a very ample line bundle on Y, and $dim(Y) \geq 4$. Let*

$$X = div(f) \subset Y$$

where $f \in H^0(\mathcal{L}^h)$ is generic with an h. Let

$$c : \mathbf{P}^1 \to C \subset X \subset Y$$

be a birational morphism onto an irreducible rational curve C. If c is free on Y, then c has unobstructed deformation on X, i.e.

$$H^1(N_{c/X}) = 0. \tag{4.4}$$

Proof. of Theorem 4.3: By the assumption of the theorem, we have a polarization of Y such that

$$Y \subset \mathbf{P}^n \tag{4.5}$$

is a smooth subvariety of dimension ≥ 4, and $\mathcal{L} = \mathcal{O}_{\mathbf{P}^n}(1)|_Y$. Let

$$s \in H^0(\mathcal{O}_{\mathbf{P}^n}(h)) \tag{4.6}$$

be generic. Let $f = s|_Y \in H^0(\mathcal{L}^h)$.

We denote

$$div(f) = X, div(s) = Z. \tag{4.7}$$

By the genericity of Z, scheme-theoretically

$$X = Y \cap Z. \tag{4.8}$$

Let $c : \mathbf{P}^1 \to X$ be generic in $Hom_{bir}(\mathbf{P}^1, X)$.

We have a non-commutative diagram of exact sequences

$$\begin{array}{cccccccccc} & & & & 0 & & 0 & & & \\ & & & & \downarrow & & \downarrow & & & \\ 0 & \rightarrow & H^0(c^*T_X) & \rightarrow & H^0(c^*T_Y) & \stackrel{\mu_1}{\rightarrow} & H^0(c^*N_{X/Y}) & \stackrel{\mu_2}{\rightarrow} & H^1(c^*T_X) & \rightarrow & H^1(c^*T_Y) \\ & & & & \downarrow & & \wr\| & & & \\ 0 & \rightarrow & H^0(c^*T_Z) & \rightarrow & H^0(c^*(T_{\mathbf{P}^n})) & \stackrel{\mu_3}{\rightarrow} & H^0(c^*(N_{Z/\mathbf{P}^n})) & \rightarrow & H^1(c^*(T_Z)) & \\ & & & & \downarrow & & \| & & & \\ & & & & H^0(c^*(N_{Y/\mathbf{P}^n})) & \stackrel{\mu_4}{\rightarrow} & H^0(c^*(N_{Z/\mathbf{P}^n})) & & & \\ & & & & \downarrow & & \downarrow & & & \\ & & & & H^1(c^*(T_Y)) & & 0 & & & \end{array} . \tag{4.9}$$

Let's define and analyze the diagram. Including all zero spaces, there are 5 rows and 6 columns. Second and third rows are parts of the long exact sequences from the short exact sequences of sheaves over $\mathbf{P}^1$,

$$0 \rightarrow c^*(T_X) \rightarrow c^*(T_Y) \rightarrow c^*(N_{X/Y}) \rightarrow 0, \tag{4.10}$$

$$0 \rightarrow c^*(T_Z) \rightarrow c^*(T_{\mathbf{P}^n}) \rightarrow c^*(N_{Z/\mathbf{P}^n}) \rightarrow 0. \tag{4.11}$$

The third column is the part of the long sequence of the short exact sequence of sheaves over $\mathbf{P}^1$,

$$0 \rightarrow c^*(T_Y) \rightarrow c^*(T_{\mathbf{P}^n}) \rightarrow c^*(N_{Y/\mathbf{P}^n}) \rightarrow 0. \tag{4.12}$$

The isomorphism in the fourth column is from the adjuntion formula.

The homomorphism μ_4 is well-defined only if c has unobstructed deformation on Y. Let's see this in the following. Because $c : \mathbf{P}^1 \to Y$ is free, therefore c has unobstructed deformation on Y. So $H^1(c^*(T_Y)) = 0$. Therefore the third column exact sequence in (4.9) splits, i.e.

$$H^0(c^*(T_{\mathbf{P}^n})) \simeq H^0(c^*(N_{Y/\mathbf{P}^n})) \oplus H^0(c^*(T_Y)). \tag{4.13}$$

We define μ_4 to be the restriction of μ_3 to the subspace, $H^0(c^*(N_{Y/\mathbf{P}^n}))$ i.e.

$$\begin{array}{rcl} \mu_4 : H^0(c^*(N_{Y/\mathbf{P}^n})) & \to & H^0(c^*(N_{Z/\mathbf{P}^n})) \\ \alpha & \to & \alpha|_t + c^*(T_Z)|_t. \end{array} \tag{4.14}$$

(But μ_4 may not be zero.). In the sense of this splitting, the map μ_3 also splits as

$$\mu_3 = \mu_1 \oplus \mu_4. \tag{4.15}$$

Next we go further to use a construction to prove that μ_4 is the zero map due to the global generation. This is just a specification of

$$H^0(c^*(N_{Y/\mathbf{P}^n}))$$

inside of

$$H^0(c^*(T_Y)).$$

By (4.13), there are a Zariski open set $U \subset \mathbf{P}^1$ and a trivial subbundle

$$E \subset c^*(N_{Y/\mathbf{P}^n})$$

over U such that

$$E \oplus c^*(T_Y)|_U = c^*(T_{\mathbf{P}^n})|_U \tag{4.16}$$

and

$$E \subset T_Z|_U. \tag{4.17}$$

Let

$$B = \{\sigma \in H^0(c^*(T_{\mathbf{P}^n})) : \sigma|_U \in E\}. \tag{4.18}$$

So B is a subspace of $H^0(c^*(T_{\mathbf{P}^n}))$. It generates a sheaf $E_{\mathbf{P}^1}$ over $\mathbf{P}^1$ (in general B could be zero, so did $E_{\mathbf{P}^1}$.). Because

$$c^*(T_Y), c^*(T_{\mathbf{P}^n})$$

are generated by global section, the sheaf $E_{\mathbf{P}^1}$ over $\mathbf{P}^1$ satisfies

$$B \simeq H^0(c^*(N_{Y/\mathbf{P}^n})). \tag{4.19}$$

and (4.17) extends to

$$E_{\mathbf{P}^1} \subset T_Z. \tag{4.20}$$

We then define μ_4 to be the restriction of μ_3 to B (this is the same as the definition before. But this time, the subspace $H^0(c^*(N_{Y/\mathbf{P}^n}))$ is uniquely identified.). By the condition (4.20), μ_4 is the zero map.

Thus

$$Image(\mu_1) = Image(\mu_3). \tag{4.21}$$

Now we consider the third row, an exact sequence in (4.9). Since Z is a generic hypersurface of $\mathbf{P}^n$, we can apply Theorem 1.1, [8] (or [7]) to obtain that $H^1(c^*(T_Z)) = 0$. The exactness of the sequence implies that μ_3 is surjective. So is μ_1. Next we shift the focus to the second row in (4.9). We apply $H^1(c^*(T_Y)) = 0$ again to obtain that $H^1(c^*(T_X)) = 0$. Using the exact sequence

$$\begin{array}{ccccccccc} 0 & \rightarrow & T_{\mathbf{P}^1} & \rightarrow & c^*(T_X) & \rightarrow & N_{c/X} & \rightarrow & 0. \end{array} \tag{4.22}$$

we obtain that

$$H^1(c^*(T_X)) = H^1(N_{c/X}) = 0. \tag{4.23}$$

$\square$

4.2 Free morphism

In this subsection, we continue the existence result — the existence of irreducible rational curves of each degree d on a generic complete intersection Calabi-Yau 3-folds. From now on we change Y to be a complete intersection as follows,

$$Y = \cap_{i=1}^{r-1} div(g_i) \tag{4.24}$$

for the fixed set of generic g_i. So Y, which is equal to X_4, is a smooth 4-dimensional complete intersection and Fano. As before for the fixed set of g_i,

$$X_3 = \cap_{i=1}^{r} div(g_i) \tag{4.25}$$

is a Calabi-Yau 3-fold contained in the Fano 4-fold $Y = X_4$.

We would like to show a generic rational curve c on X_3 is free on Y, in another word, such rational curves cover Y. Thus we assume

Assumption 4.4. *$c_0 \in Hom_{bir}(\mathbf{P}^1, X_3)$ of $deg(c_0(\mathbf{P}^1)) = d$ exists.*

We'll work in a neighborhood of each component of the incidence schemes around c_0.

Let's resume all the notations in section 2. We let $g_1, \cdots, g_{r-1}$ be fixed (where $r = n - 3$). Then the incidence scheme I_{r-1} is also fixed. Let $\mathcal{I}_{r-1}$ be a component of I_{r-1} containing c_0. Let Γ be an irreducible component of the scheme

$$\{(c, g_r) \in \mathcal{I}_{r-1} \times H^0(\mathcal{O}_{\mathbf{P}^n}(h_r)) : c^*(g_r) = 0\} \tag{4.26}$$

containing c_0. Let

$$\pi : \mathcal{I}_{r-1} \times H^0(\mathcal{O}_{\mathbf{P}^n}(h_r)) \rightarrow M_d \tag{4.27}$$

be the projection. Let $\mathcal{I} = \pi(\Gamma)$. So

$$\mathcal{I} \subset \mathcal{I}_{r-1}.$$

Let $0 \in \mathbf{P}^1$ be a generic point. Let $\mathcal{R}$ be the closure of the open scheme,

$$\mathring{\mathcal{R}} = \{(c, y) \in \mathbf{P}(\mathcal{I}) \times Y : y = c(0)\}. \tag{4.28}$$

The equation $y = c(0)$ means that c is regular at 0.

We would like to show that

Proposition 4.5. *The open scheme $\mathring{\mathcal{R}}$ is a rational map from $\mathbf{P}(\mathcal{I})$ to Y and dominates Y.*

Proof. Let l be an integer satisfying

$$0 \leq l \leq r. \tag{4.29}$$

Let $\mathcal{I}_l$ be a component of I_l containing c_0. Define

$$\mathcal{R}_l \subset \mathbf{P}(\mathcal{I}_l) \times X_{n-l} \tag{4.30}$$

to be the closure of

$$\mathring{R}_l = \{(c, x) : c(0) = x\}. \tag{4.31}$$

for each l.

First we consider the case $l < r - 1$. Suppose $\mathcal{R}_l$ is onto X_{n-l}, where $dim(X_{n-l}) = n - l$. Let $x \in X_{n-l}$ have coordinates $[0, \cdots, 0, 1]$. Then the dimension of the fibre $(\mathcal{R}_l)_x$ of $\mathcal{R}_l$ over the generic point $x \in X_{n-l}$ satisfies

$$dim((\mathcal{R}_l)_x) \geq (h_{l+1} + \cdots + h_r)d. \tag{4.32}$$

Hence after imposing $h_{l+1}d + 1$ equations,

$$g_{l+1}(c(t_1^{l+1})) = \cdots = g_{l+1}(c(t_{h_{l+1}d+1}^{l+1}) = 0,$$

we obtain that

$$dim((\mathcal{R}_{l+1})_x) \geq (h_{l+2} + \cdots + h_r)d - 1 \geq 0. \tag{4.33}$$

for all $x \in X_{n-l-1}$. Let

$$Proj : \mathbf{P}(I_l) \to \mathbf{P}^n$$

be the projection. Then the correspondence $\mathcal{R}_{l+1}$ sends $Proj(\mathcal{R}_{l+1})$ onto X_{n-l-1} (not a map). Next we show R_{l+1} is a rational map.

Applying Assumption 4.4, there is a

$$c_0 \in Proj(\mathcal{R}_{l+1}) \cap M_{bir,d}.$$

Then generic $c \in Proj(\mathcal{R}_{l+1})$ must be in $M_{bir,d}$ because $M_{bir,d}$ is an open set. Hence the correspondence $\mathcal{R}_{l+1}$ is a rational map. By the induction on the index l for

$$l = 0 \Rightarrow l = r - 2,$$

we showed that the irreducible component

$$\mathring{\mathcal{R}}_{r-1}$$

dominates Y, through the rational evaluation map

$$c \to c(0).$$

To prove Proposition 4.5 it suffices to extend above proof to $l = r$, but the situation is slightly different.

Let

$$\mathbf{t}^r = (t_1^r, \cdots, t_{h_r d+1}^r) \in sym^{h_r d+1}(\mathbf{P}^1) \tag{4.34}$$

be generic. By the result of above argument, $\mathring{\mathcal{R}}_{r-1}$ dominates Y. So for a generic $y \in Y$,

$$dim((\mathcal{R}_{r-1})_y) \geq h_r d. \tag{4.35}$$

For the fixed y, we define $\mathcal{S}_y$ to be closure of

$$\begin{aligned} \mathring{\mathcal{S}}_y \subset \mathcal{I}_{r-1} \times H^0(\mathring{\mathcal{O}_{\mathbf{P}^n}}(h_r)) \\ \mathring{\mathcal{S}}_y = \{(c, g_r) : c \in I_{r-1}, g_r(c(t_1^r)) = \cdots = g_r(c(t_{h_r d+1}^r)) = 0\} \end{aligned} \tag{4.36}$$

where $H^0(\mathring{\mathcal{O}_{\mathbf{P}^n}}(h_r))$ is an open set of space of hypersurface g_r satisfying that the Jacobian matrix of the functions in c

$$g_r(c(t_1^r)), \cdots, g_r(c(t_{h_r d+1}^r))$$

at $c \in I_{r-1}$ has full rank. By the inequality (4.35), we count the dimension to obtain the $\mathcal{S}_y$ is non-empty. Hence

$$\pi(\mathcal{S}_y) \tag{4.37}$$

is non-empty for generic y. This shows that the correspondence $\mathcal{R}$ is onto Y. Now let's show it is a rational map. Notice a generic $c \in \pi(\mathcal{S}_y)$ is a generic element in

$$\cup_{y \in Y} \pi(\mathcal{S}) = \mathcal{I}. \tag{4.38}$$

Hence it suffices to show there is one regular point for $\mathcal{R}$. By Assumption 4.4, there is a $c_0 \in \mathcal{S}_y$ such that c_0 is in $M_{bir,d}$, and $c_0(0) = y$ for some $y \in Y$ (not all y). Therefore $\mathcal{R}$ is regular at ONE point. With the same reason as above, $\mathcal{R}$ is a rational map. Thus $\mathring{\mathcal{R}}$ dominates Y. This completes the proof. □

Corollary 4.6. *Let $(c, y) \in \mathcal{R}$ be generic. Then c is a free morphism on Y.*

Proof. By Proposition 4.5, there is a Zariski open set U of $\mathcal{R}$, such that the projection

$$Proj : U \to Proj(U) \subset Y$$

is smooth. Hence its differential is onto. Let $(c, y) \in U$. Then the pull-back of the tangent bundle

$$c^*(T_Y) \simeq \oplus_k \mathcal{O}_{\mathbf{P}^1}(k) \tag{4.39}$$

does not have negative summand, i.e. $k \geq 0$. Hence a c gives a free morphism $\mathbf{P}^1 \to Y$. □

4.3 Unobstructed deformation

Notice that there is a degree d rational curve $c \in Hom_{bir}(\mathbf{P}^1, X_3)$ for a generic X_3. By corollary 4.6, it is a free morphism in Y, we apply Theorem 4.3. Then the second row of (4.9) implies a recursive formula

$$dim(H^0(c^*(X_{l+1}))) = dim(H^0(c^*(X_l))) - h_l d - 1. \tag{4.40}$$

Since

$$dim(H^0(c^*(X_0))) = (n+1)d + n,$$

$$dim(H^0(c^*(X_3))) = (n+1)d + n - \sum_{k=1}^{r} h_k d - (n-3) = 3. \tag{4.41}$$

Now we consider it from a different point of view. Because c is a birational map to its image, there are finitely many points $t_i \in \mathbf{P}^1$ where the differential map

$$c_* : T_{t_i}\mathbf{P}^1 \quad \to \quad T_{c(t_i)}Y$$

is a zero map. Assume its vanishing order at t_i is m_i . Let

$$m = \sum_i m_i. \tag{4.42}$$

Let $s(t) \in H^0(\mathcal{O}_{\mathbf{P}^1}(m))$ such that

$$div(s(t)) = \Sigma_i m_i t_i.$$

The sheaf morphism c_* is injective and induces a composed morphism ξ_s of sheaves

$$T_{\mathbf{P}^1} \stackrel{c_*}{\to} c^*(T_{X_3}) \stackrel{\frac{1}{s(t)}}{\to} c^*(T_{X_3}) \otimes \mathcal{O}_{\mathbf{P}^1}(-m). \tag{4.43}$$

It is easy to see that the induced bundle morphism ξ_s is injective. Let

$$N_m = \frac{c^*(T_{X_3}) \otimes \mathcal{O}_{\mathbf{P}^1}(-m)}{\xi_s(T_{\mathbf{P}^1})}. \tag{4.44}$$

Then

$$dim(H^0(N_m)) = dim(H^0(c^*(T_{X_3}) \otimes \mathcal{O}_{\mathbf{P}^1}(-m))) - 3. \tag{4.45}$$

On the other hand, three dimensional automorphism group of $\mathbf{P}^1$ gives a rise to a 3-dimensional subspace K of

$$H^0(c^*(T_{X_3})).$$

By (4.41), $K = H^0(c^*(T_{X_3}))$. Over each point $t \in \mathbf{P}^1$, K spans a one dimensional subspace. Hence

$$c^*(T_{X_3}) \simeq \mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(-k_1) \oplus \mathcal{O}_{\mathbf{P}^1}(-k_2), \tag{4.46}$$

where k_1, k_2 are some positive integers. This implies that

$$dim\Bigg(H^0(c^*T_{X_3} \otimes \mathcal{O}_{\mathbf{P}^1}(-m))\Bigg) = dim(H^0(\mathcal{O}_{\mathbf{P}^1}(2-m)). \tag{4.47}$$

Then

$$dim\Bigg(H^0(c^*T_{X_3} \otimes \mathcal{O}_{\mathbf{P}^1}(-m))\Bigg) = 3 - m. \tag{4.48}$$

Since $dim(H^0(N_m)) \geq 0$, by the formula (4.45), $-m \geq 0$. By the definition of m, $m = 0$. Hence c is an immersion.

Now N_{c/X_3} is a vector bundle. By the the Calabi-Yau condition,

$$N_{c/X_3} \simeq \mathcal{O}_{\mathbf{P}^1}(k) \oplus \mathcal{O}_{\mathbf{P}^1}(-2-k) \tag{4.49}$$

where k is non-positive. By Theorem 4.3, $H^1(N_{c/X_3}) = 0$. Hence $k = -1$. Since X_3 is generic, the same proof is valid for all $c \in Hom_{bir}(\mathbf{P}^1, X_3)$. Together with the existence in section 3, we complete the proof of Theorem 1.1.

A Vandermonde type of matrices

In the appendix, we give proofs of lemmas dealing with Vandermonde type of matrices that are products of diagonal matrices and Vandermonde matrices. The main purpose is to use rather elementary technique to glue matrices of smaller sizes, the Vandermonde type of matrices, to obtain a matrix of larger size (which is a Jacobian matrix of a complete intersection). We assume that the linear algebra has the ground field $\mathbb{C}$.

First we introduce and study Vandermonde type of matrices, that are smaller matrices as entries in the large block matrices. Let

$$h \in H^0(\mathcal{O}_{\mathbf{P}^1}(v)) \tag{A.1}$$

be a polynomial, where $t \in \mathbf{P}^1$ is the variable, and Let

$$\mathbf{t} = (t_1, \cdots, t_u) \in Sym^u(\mathbf{P}^1).$$

In the following in order to define matrices, we use the affine expression in subsection 2.1. Let $\mathcal{D}_u$ be a diagonal matrix

$$\mathcal{D}_u = \begin{pmatrix} h(t_1) & 0 & \cdots & 0 \\ 0 & h(t_2) & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & 0 & h(t_u) \end{pmatrix} \tag{A.2}$$

Definition A.1.

$$\mathcal{V}_0(h, \mathbf{t}, m) = \mathcal{D}_u \begin{pmatrix} t_1^m & \cdots & t_1 \\ \vdots & \cdots & \vdots \\ t_u^m & \cdots & t_u \end{pmatrix}. \tag{A.3}$$

and

$$\mathcal{V}_1(h, \mathbf{t}, m) = \mathcal{D}_u \begin{pmatrix} t_1^m & \cdots & 1 \\ \vdots & \cdots & \vdots \\ t_u^m & \cdots & 1 \end{pmatrix} \tag{A.4}$$

We call them Vandermonde type of matrices of orders 0 *and* 1 *respectively.*

Remark The Jacobian matrices J_l of incidence schemes (we are interested in) are all made of Vandermonde type of matrices.

Lemma A.2. *If h_1, h_2 are two relatively prime polynomials with distinct zeros, and degrees $\geq d$, then for generic*

$$(\mathbf{t}^1, \mathbf{t}^2) \in Sym^d(\mathbf{P}^1) \times Sym^{d+1}(\mathbf{P}^1)$$

the square matrix

$$B = \begin{pmatrix} \mathcal{V}_0(h_1, \mathbf{t}^1, d) & \mathcal{V}_1(h_2, \mathbf{t}^1, d) \\ \mathcal{V}_1(h_1, \mathbf{t}^2, d) & \mathcal{V}_0(h_2, \mathbf{t}^2, d) \end{pmatrix} \tag{A.5}$$

is non-degenerate.

Proof. It suffices to prove it for a special $\mathbf{t}^i$. Since h_1 has degree $\geq d$ and it is relatively prime to h_2, we can choose $t_1^1, \cdots, t_d^1$ to be the zeros of $h_1(t)$ and $h_2(t)$ is non-zero at all points of $\mathbf{t}^1, \mathbf{t}^2$. Then

$$B = \begin{pmatrix} 0 & \mathcal{V}_1(h_2, \mathbf{t}^1, d) \\ \mathcal{V}_1(h_1, \mathbf{t}^2, d) & \mathcal{V}_0(h_2, \mathbf{t}^2, d) \end{pmatrix} \tag{A.6}$$

Then its determinant is

$$-|\mathcal{V}_1(h_2, \mathbf{t}^1)||\mathcal{V}_1(h_1, \mathbf{t}^2)|. \tag{A.7}$$

Since $\mathcal{V}_1(h_2, \mathbf{t}^1), \mathcal{V}_1(h_1, \mathbf{t}^2)$ are types of Vandermonde matrices, the distinct t_j^i and nonvanishing h_1, h_2 imply the non-degeneracy. Therefore B is non-degenerate.

□

Lemma A.3. *Let $h_3 \in H^0(\mathcal{O}_{\mathbf{P}^1}(2d))$ and $h_1, h_2 \in H^0(\mathcal{O}_{\mathbf{P}^1}(d))$ be two vectors not on the same line through the origin. They are also pair wisely relatively prime, and all zeros are distinct. Let*

$$(\mathbf{t}^1, \mathbf{t}^2, \mathbf{t}^3) \in sym^d(\mathbf{P}^1) \times sym^d(\mathbf{P}^1) \times sym^{d+1}(\mathbf{P}^1)$$

be generic. Then the square matrix ($(3d+1)\times(3d+1)$),

$$B = \begin{pmatrix} \mathcal{V}_0(h_1, \mathbf{t}^1, d) & \mathcal{V}_0(h_2, \mathbf{t}^1, d) & \mathcal{V}_1(h_3, \mathbf{t}^1, d) \\ \mathcal{V}_0(h_1, \mathbf{t}^2, d) & \mathcal{V}_0(h_2, \mathbf{t}^2, d) & \mathcal{V}_1(h_3, \mathbf{t}^2, d) \\ \mathcal{V}_0(h_1, \mathbf{t}^3, d) & \mathcal{V}_0(h_2, \mathbf{t}^3, d) & \mathcal{V}_1(h_3, \mathbf{t}^3, d) \end{pmatrix} \tag{A.8}$$

is non-degenerate.

Proof. By the linear algebra in Proposition 2.10, [7], we obtain that the matrix $\mathcal{V}$,

$$\mathcal{V} = \begin{pmatrix} \mathcal{V}_0(h_1, \mathbf{t}^1, d) & \mathcal{V}_0(h_2, \mathbf{t}^1, d) \\ \mathcal{V}_0(h_1, \mathbf{t}^2, d) & \mathcal{V}_0(h_2, \mathbf{t}^2, d) \end{pmatrix} \tag{A.9}$$

has full rank for any $(\mathbf{t}^1, \mathbf{t}^2)$ with $2d$ distinct points in $\mathbf{P}^1$.

Then we rewrite B as a block matrix,

$$B = \begin{pmatrix} \mathcal{V} & \mathcal{A} \\ \mathcal{D} & \mathcal{V}_1(h_3, \mathbf{t}^3, d) \end{pmatrix} \tag{A.10}$$

where

$$\mathcal{D} = \begin{pmatrix} \mathcal{V}_0(h_1, \mathbf{t}^3, d) & \mathcal{V}_0(h_2, \mathbf{t}^3, d) \end{pmatrix} \tag{A.11}$$

and

$$\mathcal{A} = \begin{pmatrix} \mathcal{V}_1(h_3, \mathbf{t}^1, d) \\ \mathcal{V}_1(h_3, \mathbf{t}^2, d) \end{pmatrix}. \tag{A.12}$$

Then for generic $\mathbf{t}^1, \mathbf{t}^2, \mathbf{t}^3$, B is column-equivalent to

$$\begin{pmatrix} \mathcal{V} & 0 \\ \mathcal{D} & -\mathcal{D}\mathcal{V}^{-1}\mathcal{A} + \mathcal{V}_1(h_3, \mathbf{t}^3, d) \end{pmatrix}. \tag{A.13}$$

By choosing the points of $\mathbf{t}^1, \mathbf{t}^2$ to be the zeros of h_3 and $\mathcal{V}$ remains invertible, we obtain that

$$\mathcal{A} = 0.$$

Thus we have a specialization

$$-\mathcal{D}\mathcal{V}^{-1}\mathcal{A} + \mathcal{V}_1(h_3, \mathbf{t}^3, d) = \mathcal{V}_1(h_3, \mathbf{t}^3, d)$$

which is non-degenerate. Thus for generic $\mathbf{t}^1, \mathbf{t}^2, \mathbf{t}^3$,

$$-\mathcal{D}\mathcal{V}^{-1}\mathcal{A} + \mathcal{V}_1(h_3, \mathbf{t}^3, d)$$

is non-degenerate, so is B.

$\square$

We'll use two different types matrices B in Lemmas A. 2, A.3 as building blocks to obtain matrices of larger sizes. We'll show that non-degeneracy of large matrices is the result of that of the smaller, block matrices.

Lemma A.4. *Let $B_{ij}, i = 1, 2, j = 1, \cdots, 6$ be $(3d+1) \times (d+1)$ matrices. Assume*

(1)

$$\begin{pmatrix} B_{13} & B_{14} \end{pmatrix} = \begin{pmatrix} B_{23} & B_{24} \end{pmatrix} \tag{A.14}$$

(2)

$$\begin{pmatrix} B_{13} & B_{14} & B_{16} \end{pmatrix} \tag{A.15}$$

and

$$\begin{pmatrix} B_{11} & B_{12} - B_{22} & B_{25} \end{pmatrix} \tag{A.16}$$

have full rank,

(3) the columns of

$$\begin{pmatrix} B_{13} & B_{14} \end{pmatrix} \tag{A.17}$$

span the columns of

$$\begin{pmatrix} B_{11} & B_{12} \end{pmatrix} \tag{A.18}$$

Then

$$J = \begin{pmatrix} B_{11} & B_{12} & B_{13} & B_{14} & 0 & B_{16} \\ 0 & B_{22} & B_{23} & B_{24} & B_{25} & B_{26} \end{pmatrix} \tag{A.19}$$

is non-degenerate.

Proof. In the following, we apply the column and row operations to the matrix. They will not change its rank. Applying the row operations on the matrix J we obtain it is row-equivalent to

$$J = \begin{pmatrix} B_{11} & B_{12} & B_{13} & B_{14} & 0 & B_{16} \\ -B_{11} & B_{22} - B_{12} & 0 & 0 & B_{25} & B_{26} - B_{16} \end{pmatrix} \tag{A.20}$$

By the condition (3), it is column-equivalent to

$$J = \begin{pmatrix} 0 & 0 & B_{13} & B_{14} & 0 & B_{16} \\ -B_{11} & B_{22} - B_{12} & 0 & 0 & B_{25} & B_{26} - B_{16} \end{pmatrix} \tag{A.21}$$

By the condition (2), it is further column equivalent to

$$J = \begin{pmatrix} 0 & 0 & B_{13} & B_{14} & 0 & B_{16} \\ -B_{11} & B_{22} - B_{12} & 0 & 0 & B_{25} & 0 \end{pmatrix} \tag{A.22}$$

By the condition (2), we obtain it has full rank.

□

Lemma A.5. *Consider the block matrix*

$$J = \begin{pmatrix} B_{11} & B_{12} & B_{13} & B_{14} & B_{15} & 0 & 0 \\ 0 & 0 & B_{23} & B_{24} & B_{25} & B_{26} & 0 \\ 0 & 0 & 0 & B_{34} & B_{35} & B_{36} & B_{37} \end{pmatrix} \tag{A.23}$$

where the entries in first row are $(3d+1) \times (d+1)$ *matrices and all the rest are* $(2d+1) \times (d+1)$ *matrices. We assume*

(1)

$$\begin{pmatrix} B_{24} & B_{25} & B_{26} \end{pmatrix} = \begin{pmatrix} B_{34} & B_{35} & B_{36} \end{pmatrix}. \tag{A.24}$$

(2)

$$\begin{pmatrix} B_{11} & B_{12} & B_{15} \end{pmatrix} \tag{A.25}$$

,

$$\begin{pmatrix} B_{23} & -B_{37} \end{pmatrix} \tag{A.26}$$

and

$$\begin{pmatrix} B_{34} & B_{36} \end{pmatrix} \tag{A.27}$$

have full rank,

(3) the columns of

$$\begin{pmatrix} B_{11} & B_{12} \end{pmatrix} \tag{A.28}$$

span the columns of

$$\begin{pmatrix} B_{13} & B_{14} \end{pmatrix}, \tag{A.29}$$

then the matrix J *has full rank.*

Proof. The proof uses row and column operations which are the same as those in Lemma A.4. So in the following we just list the equivalent matrices in the reduction.

$$J = \begin{pmatrix} B_{11} & B_{12} & B_{13} & B_{14} & B_{15} & 0 & 0 \\ 0 & 0 & B_{23} & B_{24} & B_{25} & B_{26} & 0 \\ 0 & 0 & 0 & B_{34} & B_{35} & B_{36} & B_{37} \end{pmatrix} \tag{A.30}$$

$$\Downarrow_{row\ operations}$$

$$\begin{pmatrix} B_{11} & B_{12} & B_{13} & B_{14} & B_{15} & 0 & 0 \\ 0 & 0 & B_{23} & 0 & 0 & 0 & -B_{37} \\ 0 & 0 & 0 & B_{34} & B_{35} & B_{36} & B_{37} \end{pmatrix} \tag{A.31}$$

$$\Downarrow \textit{column operations}$$

$$J = \begin{pmatrix} B_{11} & B_{12} & 0 & 0 & B_{15} & 0 & 0 \\ 0 & 0 & B_{23} & 0 & 0 & 0 & -B_{37} \\ 0 & 0 & 0 & B_{34} & B_{35} & B_{36} & B_{37} \end{pmatrix} \tag{A.32}$$

$$\Downarrow \textit{column operations}$$

$$J = \begin{pmatrix} B_{11} & B_{12} & 0 & 0 & B_{15} & 0 & 0 \\ 0 & 0 & B_{23} & 0 & 0 & 0 & -B_{37} \\ 0 & 0 & 0 & B_{34} & 0 & B_{36} & 0 \end{pmatrix}. \tag{A.33}$$

The last matrix has full rank.

□

Lemma A.6. *Consider the block matrix*

$$J = \begin{pmatrix} B_{11} & B_{12} & 0 & 0 \\ 0 & B_{22} & B_{23} & 0 \\ 0 & 0 & B_{33} & B_{34} \\ 0 & B_{42} & 0 & B_{44} \end{pmatrix}, \tag{A.34}$$

where B_{ij} are non degenerate square matrices of the same size. If

$$\begin{pmatrix} -B_{22}B_{23}^{-1}B_{33} & B_{34} \\ B_{42} & B_{44} \end{pmatrix} \tag{A.35}$$

is non-degenerate, so is J.

Proof. As before we perform row and column operations on J to obtain

$$\begin{pmatrix} B_{11} & 0 & 0 & 0 \\ 0 & B_{22} & B_{23} & 0 \\ 0 & 0 & B_{33} & B_{34} \\ 0 & B_{42} & 0 & B_{44} \end{pmatrix} \tag{A.36}$$

$$\Downarrow$$

$$\begin{pmatrix} B_{11} & 0 & 0 & 0 \\ 0 & 0 & B_{23} & 0 \\ 0 & -B_{22}B_{23}^{-1}B_{33} & B_{33} & B_{34} \\ 0 & B_{42} & 0 & B_{44} \end{pmatrix} \tag{A.37}$$

$$\Downarrow$$

$$\begin{pmatrix} B_{11} & 0 & 0 & 0 \\ 0 & 0 & B_{23} & 0 \\ 0 & -B_{22}B_{23}^{-1}B_{33} & 0 & B_{34} \\ 0 & B_{42} & 0 & B_{44} \end{pmatrix} \tag{A.38}$$

By the assumption,

$$\begin{pmatrix} -B_{22}B_{23}^{-1}B_{33} & B_{34} \\ B_{42} & B_{44} \end{pmatrix} \tag{A.39}$$

is non-degenerate, so is J.

□

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