Grothendieck duality

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Abstract

Let X be a complex projective manifold of dimension n. The total sum of singular cohomology groups $H^r(X;\mathbb{Q})$ in $\mathbb{Q}$ with all degrees, denoted by $H(X;\mathbb{Q})$ has the filtration by the level

$$H(X;\mathbb{Q}) = \mathcal{N}_n H(X) \supset \mathcal{N}_{n-1} H(X) \supset \cdots \supset \mathcal{N}_1 H(X) \supset \mathcal{N}_0 H(X) \quad (0.1)$$

where

$$\mathcal{N}_k H(X) = \sum_{i=0}^{n-[\frac{k}{2}]} N^i H^{2i+k}(X)$$

and $N^i H^{2i+k}(X)$ is the subgroup in the coniveau filtration with the coniveau i and level k. On the other hand, the hard Lefschezt theorem yields the automorphism of Lefschetz type,

$$\begin{array}{rcl} \mathcal{P} : H(X;\mathbb{Q}) & \rightarrow & H(X;\mathbb{Q}) \\ \left(\begin{array}{c} \alpha \\ for\ deg(\alpha) \leq n \end{array} \right. & \rightarrow & \left. \alpha \cdot (power\ of\ hyperplane) \right) \end{array}$$

which in topology is the Poincaré duality. In this paper, we show $\mathcal{P}$ preserves the filtration (0.1), i.e. $\mathcal{P}$ maps the subgroup $\mathcal{N}_k H(X)$ onto itself.

Contents

1 Introduction

1.1 Statement

We study subgroups of cohomology groups. It starts with the Lefschetz standard conjecture ([3]). The conjecture addresses the etál cohomology of a smooth projective variety X of dimension n over any algebraically closed field. We step back to assume the ground field is $\mathbb{C}$, and replace the etál cohomology with the singular cohomology $H(X;\mathbb{Q})$ with rational coefficients. Additional superscript denotes the degree of the cohomology. Let u be the hyperplane section class in $H^2(X;\mathbb{Z})$. For $0\leq h\leq n$, let L^h denote the Lefschetz homomorphism on the cohomology

$$\begin{array}{ccc} L^h : \sum_{i=0}^{2n-2h} H^i(X;\mathbb{Q}) & \rightarrow & \sum_{i=0}^{2n-2h} H^{i+2h}(X;\mathbb{Q}) \\ \alpha & \rightarrow & \alpha\cdot u^h. \end{array} \tag{1.1}$$

The hard Lefschetz theorem asserts L^h is restricted to an isomorphism between $H^{n-h}(X;\mathbb{Q})$ and $H^{n+h}(X;\mathbb{Q})$. Grothendieck envisioned that L^h is arithmetic in nature. He proposed the Lefschetz standard conjecture*:

Conjecture 1.1. *Let $A^j(X)\subset H^{2j}(X;\mathbb{Q})$ for a non-negative integer j be the image of the cycle map. Then the restriction L_0^{q-p} of L^{q-p} to $A^p(X)$,*

$$\begin{array}{ccc} L_0^{q-p} : A^p(X) & \rightarrow & A^q(X) \\ \alpha & \rightarrow & \alpha\cdot u^{q-p}. \end{array} \tag{1.2}$$

for $p+q=n, q\geq p$ is an isomorphism.

We extend the conjecture to all subgroups of the same type. Let

$$N^i H^{2i+k}(X)=\mathcal{N}_k H(X)\cap H^{2i+k}(X;\mathbb{Q})$$

be the subspaces in the coniveau filtration of the cohomology. They will be called the coniveau subgroups. Since $A^j(X)=N^jH^{2j}(X)$, coniveau subgroups can be viewed as an extension of the cycle group to non zero level k. For the extension, we prove the main theorem.

Key words: Cohomology, coniveau filtration, Lefschetz standard conjecture, current's intersection

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*Precisely, Grothendieck's original proposal addresses etál cohomology. Later, it is extended to all abstract Weil cohomology (which itself is inspired by Grothendieck's idea). Our result, though, only holds for singular cohomology.

Theorem 1.2. *(Main theorem)*
Let p, q, k be whole numbers satisfying

$$p + q + k = n, h = q - p, q \geq p.$$

Then the restricted Lefschetz homomorphism (1.2) to the subspace

$$\begin{array}{ccc} L_k^h : N^p H^{2p+k}(X) & \to & N^q H^{2q+k}(X) \\ \alpha & \to & \alpha \cdot u^h. \end{array} \tag{1.3}$$

is still an isomorphism. We call the family of isomorphisms L_k^h Grothendieck duality

So for $k = 0$, Main theorem is

Corollary 1.3. *Conjecture 1.1 is correct.*

The following is an example of this duality.

Example 1.4. *Let X a smooth projective variety over $\mathbb{C}$ with $dim(X) = 4$. Retain all indexes p, q, k as in Theorem 1.2. The following table classifies the Grothendieck dualities.*

row	p	q	k	Grothendieck duality
I	0	1	3	$N^0H^3(X) \overset{L_3^1}{\simeq} N^1H^5(X)$
II	0	2	2	$N^0H^2(X) \overset{L_2^2}{\simeq} N^2H^6(X)$
III	0	3	1	$N^0H^1(X) \overset{L_1^3}{\simeq} N^3H^7(X)$
IV	1	1	2	$N^1H^4(X) \overset{L_2^0}{=} N^1H^4(X)$
V	1	2	1	$N^1H^3(X) \overset{L_1^1}{\simeq} N^2H^5(X)$
VI	1	3	0	$N^1H^2(X) \overset{L_0^2}{\simeq} N^3H^6(X)$
VII	2	2	0	$N^2H^4(X) \overset{L_0^0}{=} N^2H^4(X)$

The duality maps in the rows I, II, III form the Poincaré duality that is topological (by the hard Lefschetz theorem). The duality map in the row VI is the prediction of the Lefschetz standard conjecture (it is true by the Lefschetz $(1, 1)$ theorem). The duality maps in the rows IV, VII are the identity maps. The duality map in the row V is the content of this paper $-$ Grothendieck duality.

1.2 Outline of the proof

Let X be a complex projective manifold of dimension n, and V^h the generic h-codimensional plane section. The plane section map L_k^h factors through the local cohomology in the composite,

$$N^p H^{2p+k}(X) \quad \xrightarrow{L_{loc}^h} \quad N^q H_{V^h}^{2q+k}(X) \quad \xrightarrow{i} \quad N^q H^{2q+k}(X) \tag{1.4}$$

where the subscript V^h on the cohomology denotes the support and i is the inclusion map. The composition L_k^h is injective due to the hard Lefschetz theorem. Hence it is sufficient to prove the surjectivity of both maps. The proof of the surjectivity is the content of this paper whose principle idea is to change the main objects in cohomology theory to attack currents on differential manifolds. We explore it in three steps.

(1) Switch the focus to currents. Based on de Rham cohomology denoted also by $H(X;\mathbb{C})$, we interpret the coniveau filtration of cohomology by using C^∞ currents. Let $\mathcal{C}^p H^{2p+k}(X)$ be the subgroups of cohomology, that consists of closed currents supported on subvarieties of codimension at least p. Then in the appendix A, we prove that

$$\mathcal{C}^p H^{2p+k}(X) = N^p H^{2p+k}(X). \tag{1.5}$$

The principle leads to the study of currents.

(2) Develop a measure-theoretical theory in intersection– real intersection theory (see [4]). The theory claims that on the C^∞ manifold X, there is a subgroup $\mathscr{L}(X) \subset \mathscr{D}'(X)$ in the group of currents, called Lebesgue currents. It is sufficiently large so that it includes all C^∞ singular chains and C^∞ forms. On $\mathscr{L}(X)$, there is a bilinear homomorphism dependent of the de Rham data attached to the C^∞ structure of X,

$$\begin{array}{rcc} [\cdot \wedge \cdot] : \mathscr{L}(X) \times \mathscr{L}(X) & \rightarrow & \mathscr{L}(X) \\ (T_1, T_2) & \rightarrow & [T_1 \wedge T_2], \end{array} \tag{1.6}$$

referred to as the intersection of currents. The current's intersection is measure-theoretical, but the next step shows it can be reduced to the cup product on cohomology to give properties of the coniveau filtration.

(3) At the last, the content of this paper constructs the extrinsic geometry to practice the principles in (1) and (2). The following is this extrinsic geometry.

Let $\mathbb{A}^{n+2}$ be the affine space over $\mathbb{C}$ with the standard basis

$$\mathbf{e}_0, \cdots, \mathbf{e}_{n+1}.$$

Let h be a natural number $\leq n$. Consider two subspaces

$$\begin{array}{c} \mathbb{A}^{n+2-h} = span(\mathbf{e}_0, \cdots, \mathbf{e}_{n+1-h}), \\ \mathbb{A}^h = span(\mathbf{e}_{n+2-h}, \cdots, \mathbf{e}_{n+1}). \end{array} \tag{1.7}$$

Then

$$\mathbb{A}^{n+2-h} \oplus \mathbb{A}^h = \mathbb{A}^{n+2}. \tag{1.8}$$

Next we define a specific variation of $\mathbb{A}^h$. Let $\mathbb{C} \cup \{\infty\} \simeq \mathbf{P}^1$ be the parameter space of the variation, denoted by Υ, where ∞ is the infinity point of $\mathbf{P}^1$. The family of linear spaces parametrized by Υ is defined by

$$\mathbb{A}_z^h = span(z\mathbf{e}_{n+2-h} - \mathbf{e}_0, \mathbf{e}_{n+3-h}, \cdots, \mathbf{e}_{n+1}), \quad for\ z \in \mathbb{C} \tag{1.9}$$

and $\mathbb{A}_0^h$ is the original $\mathbb{A}^h$. So $\Upsilon \subset G(h, n+2)$. Let $U = \mathbb{C}$ be the affine open subset that parametrizes those $\mathbb{A}_z^h$ satisfying the decomposition

$$\mathbb{A}^{n+2} = \mathbb{A}^{n+2-h} \oplus \mathbb{A}_z^h. \tag{1.10}$$

The point $z = \infty$ not in U corresponds to the plane $\mathbb{A}_\infty^h$ that fails the decomposition (1.10). We call $z = \infty$ the unstable point, others stable points. Let $\mathbf{x}$ be a vector in $\mathbb{A}^{n+2}$. Therefore for each stable point $z \in U$, we have the unique decomposition

$$\mathbf{x} = \mathbf{x}_1(z) + \mathbf{x}_2(z). \tag{1.11}$$

The decomposition gives a family of linear transformations,

$$\begin{array}{ccc} \mathbb{A} \times U \times (\mathbb{A}^{n+2-h} \oplus \mathbb{A}_z^h) & \rightarrow & \mathbb{A}^{n+2-h} \oplus \mathbb{A}_z^h = \mathbb{A}^{n+2} \\ (t, z, \mathbf{x}_1(z) + \mathbf{x}_2(z)) & \rightarrow & \mathbf{x}_1(z) + t\mathbf{x}_2(z). \end{array} \tag{1.12}$$

Furthermore it gives a family of linear transformations,

$$\begin{array}{ccc} \mathbb{A} \times U \times (\mathbb{A}^{n+2-h} \oplus \mathbb{A}_z^h) & \rightarrow & \mathbb{A}^{n+2-h} \oplus \mathbb{A}_z^h = \mathbb{A}^{n+2} \\ (t, z, \mathbf{x}_1(z) + \mathbf{x}_2(z)) & \rightarrow & \mathbf{x}_1(z) + t\mathbf{x}_2(z) \end{array} \tag{1.13}$$

which yields a rational map of the projective variety

$$\begin{array}{cccc} g: & \mathbf{P}^1 \times \Upsilon \times \mathbf{P}^{n+1} & \dashrightarrow & \mathbf{P}^{n+1} \\ & (t, z, [\mathbf{x}_1(z) + \mathbf{x}_2(z)]) & \dashrightarrow & [\mathbf{x}_1(z) + t\mathbf{x}_2(z)]. \end{array}$$

$\Big($In homomogenous coordinates, g is the rational map

$$g: \mathbf{P}^1 \times \Upsilon \times \mathbf{P}^{n+1} \dashrightarrow \mathbf{P}^{n+1} \tag{1.14}$$

$$\begin{array}{c} (t, z, [x_0, \cdots, x_{n+1}]) \\ \downarrow \\ [z(1-t)x_{n+2-h} + x_0, x_1, \cdots, x_{n+h-1}, tx_{n+2-h}, \cdots, tx_{n+1}] \end{array} \tag{1.15}$$

where $x_0, \cdots, x_{n+1}$ are the homogeneous coordinates in the basis $\mathbf{e}_0, \cdots, \mathbf{e}_{n+1}$.$\Big)$.

Let

$$\Omega = graph(g) \subset \mathbf{P}^1 \times \Upsilon \times \mathbf{P}^{n+1} \times \mathbf{P}^{n+1} \tag{1.16}$$

be the graph defined to be the Zariski closure of the graph at the regular locus. Denote the projectivization $\mathbf{P}(\mathbb{A}^{n+2-h})$ by $\mathbf{P}^{n+1-h}$.

Now we consider the smooth projective variety X of dimension n, equipped with the polarization $u \in H^2(X; \mathbb{Z})$. Let

$$\mu : X \to \mathbf{P}^{n+1}$$

be a birational morphism to a hypersurface of $\mathbf{P}^{n+1}$ in a general position, in particular $\mu(X)$ is in a general position as a subvariety with respect to the polarization, X has the very ample line bundle $\mu^*(\mathcal{O}_{\mathbf{P}^{n+1}}(1))$ such that $u = c_1(\mu^*(\mathcal{O}_{\mathbf{P}^{n+1}}(1)))$ is the original polarization. The collection of the 4 items, the $\mathbf{P}^{n+1}$, its coordinates system, the linear map g, and the μ is called **cone data**. The cone data can be obtained through any embedding $X \subset \mathbf{P}^N$, by taking a projection to a generic subspace: $X \to \mathbf{P}^{n+1}$.

Notation: we use similar notations as those in intersection theory whenever they are well-defined. In particular, If $f : M_1 \to M_2$ is a C^∞ map between two compact manifolds equipped with de Rham data. Let $M_1 \times M_2$ be equipped with the product de Rham data, and $P_1 : M_1 \times M_2 \to M_1$ be the projection. For $T_i \in \mathscr{L}(M_i)$,

$$[T_1 \wedge_f T_2] := (P_1)_* \Big[graph(f) \wedge (T_1 \times T_2) \Big].$$

where $T_1 \times T_2$ is the tensor product of currents. For a test form ϕ of M_1, the evaluation $T_1(\phi)$ is denoted by the integral notation

$$\int_{T_1} \phi. \tag{1.17}$$

Le $\tau = (id, id, \mu, \mu)$ be the morphism

$$\mathbf{P}^1 \times \Upsilon \times X \times X \quad \to \quad \mathbf{P}^1 \times \Upsilon \times \mathbf{P}^{n+1} \times \mathbf{P}^{n+1}$$

and $\tau_0 = (id, \mu, \mu)$ be the morphism

$$\Upsilon \times V^h \times X \quad \to \quad \Upsilon \times \mathbf{P}^{n+1-h} \times \mathbf{P}^{n+1}$$

where $V^h = \mathbf{P}^{n+1-h} \cap X$.

Next we construct the cone family in currents. Let σ be a closed Lebesgue current in X. Denote the support of the current by $|\sigma|$. Let $\overline{|\sigma|}$ be the Zariski closure of the support (the smallest subvariety containing $|\sigma|$). We say σ is in general position if $\overline{|\sigma|}$ is. Assume $\mathbf{P}^1, \Upsilon, X$ are equipped with de Rham data and all the Cartesian products are equipped with the product de Rham data. Let $P_{14} : \mathbf{P}^1 \times \Upsilon \times X \times X \to \mathbf{P}^1 \times X \quad (1st \times 4th)$ be the projection. Then for a generally positioned σ, let

$$\psi(\sigma) := (P_{14})_* \bigg([(\mathbf{P}^1 \times \Upsilon \times \sigma \times X)] \wedge_\tau \Omega \bigg) \tag{1.18}$$

The current $\psi(\sigma)$ determines a family of currents $\psi_t(\sigma)$ (see definition 3.12 in part II, [4]). We call $\psi(\sigma)$ the cone family and $\psi_t(\sigma)$ the end current of the cone family.

Proposition 1.5. *Assume X is equipped with a de Rham data. Denote the subgroup of the closed C^∞ singular cycles by $Z(\cdot)$, and the subgroup of closed Lebesgue currents by $\mathscr{L}_C(\cdot)$. Let $\sigma \in \mathscr{L}_C(X)$ such that $dim(\sigma) + K(\sigma) < n - h$ where $K(\sigma)$ is the maximal level of the coniveau subgroup that contains the cohomology of σ in X.*

(1) $\psi_0(\sigma)$ is a current supported on V^h,

(2) $\psi_1(\sigma) \overset{hom.equi.}{\frown} \sigma$.

(3) There exists an operator, called cone operator,

$$\mathbf{Con}^h : Z(X) \quad \to \quad \mathscr{L}_C(X) \tag{1.19}$$

such that

$$[\mathbf{Con}^h(c) \wedge V^h] = c. \tag{1.20}$$

(4) If the cohomology of $c \in Z(X)$ is contained in a coniveau subgroup of level k, so is the cohomology of $\mathbf{Con}^h(c)$.

The cone family and cone operator are extrinsic and their truth is the analysis based on [4]. However, Proposition 1.5 implies the algebro-geometric result, Theorem 1.2.

Proof. of Theorem 1.2: By the Leibniz rule in II, [4], all end currents are closed. By Corollary 3.14 in II, [4], $\psi_0(\sigma), \psi_1(\sigma)$ are homological. Next we observe the level. Denote the cohomology class of a closed current $*$ by the angle bracket $\langle * \rangle$. Let σ be a singular cycle such that $\langle \sigma \rangle \in H^{2p+k}(X;\mathbb{Q})$ has the maximal level k, i.e. its cohomology class lies in the subgroup of the coniveau filtration $N^pH^{2p+k}(X)$ with the maximal level k. Then first we notice the dimension of Ω is $n+1$. Thus the degree of the cohomology of $\psi_t(\sigma)$ for all t is $2p+k$. Secondly the $\psi_t(\sigma)$ is defined by current's intersection which determines that the support $|\psi_t(\sigma)|$ lies in the algebraic set

$$P_{14}\bigg((\{t\} \times U \times \overline{|\sigma|} \times X) \cap \tau^{-1}(\Omega)\bigg). \tag{1.21}$$

If either the intersection above is not proper, or the projection is not generically 1-to-1, then $\psi_t(\sigma) = 0$ that belongs to any subgroups. So we may assume both the intersection is proper and the projection is generically 1-to-1. Then the algebraic set (1.21) for the non zero $\psi_t(\sigma)$ has codimension $p + k$ (real dimension $2p+2k$). This shows the cohomology of $\psi_t(\sigma)$ lies in the subgroup of level k. So ψ_t as an operator on cohomology preserves the level of the subgroups in the coniveau filtration. Then parts (1) and (2) imply i is surjective.

Now we let σ be a C^∞ singular cycle representing the cohomology class. The part (3) implies that $\mathbf{Con}^h(\sigma)$ represents the cohomological inverse of $\langle\sigma\rangle$. Part (4) states that the operator $\mathbf{Con}^h$ preserves the level. Hence L^h_{loc} is surjective.

□

In the next section, we give the proof of Proposition 1.5. The appendix includes the base of this paper, the currents version of the coniveau filtration, and a lemma about the iterated limits.

2 Cone family

2.1 0 end cycle

Claim 2.1. *The part (1) of Proposition 1.5 is true*

Proof. Since $\overline{|\sigma|}$ is in general position, by the supportivity of current's intersection (part II, [4]) the support of $\psi_0(\sigma)$ lies in the image of the proper morphism,

$$P_4\Big((\{0\}\times\Upsilon\times|\sigma|\times X)\cap\tau^{-1}(\Omega)\Big) \tag{2.1}$$

where $P_4 : \mathbf{P}^1\times\Upsilon\times X\times X\to X(4th)$ is the projection. Since $\mathbf{P}^h$ is generic and $dim(\sigma)+k<n-h$, $\mathbf{P}^h\cap\sigma=\varnothing$. Then according to the formula of the linear map g (1.15), all fibres of the fibration,

$$(\{0\}\times\Upsilon\times\overline{|\sigma|}\times X)\cap\tau^{-1}(\Omega^0)\to\Upsilon\times\overline{|\sigma|} \tag{2.2}$$

must lie in the plane section V^h. Over the unstable point, the fibre is in the closure, therefore still lies in V^h. Hence the projection

$$P_4\Big((\{0\}\times\Upsilon\times\overline{|\sigma|}\times X)\cap\tau^{-1}(\Omega)\Big) \tag{2.3}$$

lies in V^h. Therefore $\psi_0(\sigma)$ also lies in V^h.

□

2.2 1 end cycle

Claim 2.2. *The part (2) of Proposition 1.5 is correct.*

Proof. Let $\mathbf{P}^1,\Upsilon,X$ be equipped with de Rham data. By choosing appropriate de Rham data, we have the conditional associativity, Proposition 1.7 in II, [4] to obtain

$$[\mathbf{P}^1\times\Upsilon\times\sigma\times X)\wedge_\tau\Omega]=\Big[\mathbf{P}^1\times\Upsilon\times\sigma\times X)\wedge[\big(\mathbf{P}^1\times\Upsilon\times X\times X\big)\wedge_\tau\Omega]\Big].$$

So we focus on the current

$$[\mathbf{P}^1 \times \Upsilon \times X \times X) \wedge_\tau \Omega]. \tag{2.4}$$

By the real intersection theory,

$$[\big(\mathbf{P}^1 \times \Upsilon \times X \times X\big) \wedge_\tau \Omega]$$

is independent of de Rham data and is the current of the algebraic cycle

$$\big(\mathbf{P}^1 \times \Upsilon \times X \times X\big) \bullet_\tau \Omega. \tag{2.5}$$

The support has the trivial component

$$\{1\} \times \Upsilon \times \Delta_X \tag{2.6}$$

where Δ_X is the diagonal current of X. Let Σ be the rest which is reduced and irreducible. Since over the generic parameters t, z, the fibre of $\Omega \to \mathbf{P}^1 \times \Upsilon$ is a graph of a non-degenerated linear transformation, the intersection (2.5) is generically transversal. Hence

$$\big(\mathbf{P}^1 \times \Upsilon \times X \times X\big) \bullet_\tau \Omega = \Sigma + m\{1\} \times \Upsilon \times \Delta_\sigma. \tag{2.7}$$

where m is the intersection multiplicity. Since the second component

$$\{1\} \times \Upsilon \times \Delta_\sigma$$

is a trivial Cartesian product, the projection of the cycle to $\{1\} \times X(4th)$ has fibres of positive dimensions. Thus the cone family $\psi(\sigma)$ is equal to

$$(P_{14})_* \Big[\big(\mathbf{P}^1 \times \Upsilon \times \sigma \times X\big) \wedge \Sigma\Big]. \tag{2.8}$$

Next we see the fibre structure of Σ. Notice since X is generic with respect of the polarization, no component of the scheme intersection

$$\tau(\Sigma) \cap \tau(\mathbf{P}^1 \times \Upsilon \times X \times X)$$

lies the non-isomorphic locus of the birational map τ. Thus it suffices to work with the cycle-intersection in the $\mathbf{P}^{n+1} \times \mathbf{P}^{n+1}$. The key ingredient is the following specialization at $t = 1$ for Σ. Let $\mu(X)$ be the hypersurface of $\mathbf{P}^{n+1}$. Assume $\mu(X)$ is defined by the polynomial f. Then for $\mu^2 = (\mu, \mu)$, $\mu^2(X \times X)$ is a complete intersection defined by two polynomials $f(\mathbf{x}), f(\mathbf{y})$ in

$$\mathbf{P}^{n+1} \times \mathbf{P}^{n+1}$$

for $(\mathbf{x}, \mathbf{y}) \in \mathbf{P}^{n+1} \times \mathbf{P}^{n+1}$. Then the fibre

$$\mu^2(\Sigma_t^z),$$

over the fixed t, z for $t \neq 1$ is the fibre

$$\Omega_t^z$$

defined by two hypersurfaces $f(\mathbf{x}), f(\mathbf{y})$. As in (1.11), we let $[\mathbf{x}_1(z)], [\mathbf{x}_2(z)]$ be the points in the decomposition

$$\mathbf{P}^{n+1} = \mathbf{P}^{n+1-h} \# \mathbf{P}_z^{h-1}, \quad \textit{dependent of } z,$$

where $\mathbf{P}_z^{h-1} = \mathbf{P}(\mathbb{A}_z^h)$ and $\#$ is the linear join. Since t is near 1, it can not be 0 or ∞. Then Ω_t^z is a graph isomorphic to $\mathbf{P}^{n+1}$ expressed as the graph

$$\left\{ \left([\mathbf{x}_1(z) + \mathbf{x}_2(z)], [\mathbf{x}_1(z) + t\mathbf{x}_2(z)] \right) \right\} \subset \mathbf{P}^{n+1} \times \mathbf{P}^{n+1}.$$

Then $\mu^2(\Sigma_t^z)$ is a complete intersection explicitly defined by

$$f\Big(\mathbf{x}_1(z) + \mathbf{x}_2(z)\Big), f\Big(\mathbf{x}_1(z) + t\mathbf{x}_2(z)\Big) \tag{2.9}$$

inside of $\Omega_t^z \simeq \mathbf{P}^{n+1}$. Two polynomials are equal at $t = 1$. Thus for the specialization we consider the expansion in the factor $t - 1$,

$$\begin{aligned} & f\Big(\mathbf{x}_1(z) + \mathbf{x}_2(z)\Big) - f\Big(\mathbf{x}_1(z) + t\mathbf{x}_2(z)\Big) \\ & = (t-1)F_1^z(\mathbf{x}_1 + \mathbf{x}_2) + (t-1)^2 F_2^z(\mathbf{x}_1 + \mathbf{x}_2) + \cdots \end{aligned}$$

where $F_r^z(\mathbf{x}_1 + \mathbf{x}_2)$ is a hypersurface in

$$\{[\mathbf{x}_1 + \mathbf{x}_2]\} = \mathbf{P}^{n+1} \simeq \Delta_{\mathbf{P}^{n+1}}$$

dependent of z. By the assumption on the cone data,

$$\{F_1^z(\mathbf{x}_1 + \mathbf{x}_2) = 0\}$$

is a varied hypersurface with $z \in U$, of degree 1. Notice the inverse image of $\Delta_{\mathbf{P}^{n+1}}$ is the diagonal Δ_X of X. So the specialized fibre Σ_1 over $t = 1$ is a hypersurface in $\{1\} \times \Upsilon \times \Delta_X$ that itself is fibred over Υ such that each fibre over $z \in \Upsilon$ is a hypersurface of the diagonal Δ_X. Furthermore Σ_1 covers Δ_X with the multiplicity 1.

This shows the fibre structure of Σ_1. Now we return to the calculation. Denote the cohomology of a closed current by the angle bracket $\langle * \rangle$. The cohomology $\langle \psi_1(\sigma) \rangle$ is equal to

$$P_* \Big\langle [(\{1\} \times X) \wedge (P_{14})_* [\big(\mathbf{P}^1 \times \Upsilon \times \sigma \times X\big) \wedge \Sigma]] \Big\rangle \tag{2.10}$$

where $P : X \times X \to X(2nd)$ is the projection. Then we use the algebraic properties of the cup product to obtain

$$(2.10) = (P_3)_*\Big(\langle \Upsilon \times \sigma \times X\rangle \cup \langle \Sigma_1\rangle\Big) \tag{2.11}$$

where $P_3 : \Upsilon \times X \times X \to X(3rd)$ is the projection. Let $P_{23} : \Upsilon \times X \times X \to X \times X$ be the projection. By the fibre structure of Σ_1, the projection $(P_{23})_*(\Sigma) = \Delta_X$ as a cycle. So we apply the projection formula to (2.11) to obtain that

$$(2.11) = P_*\big(\langle \sigma \times X\rangle \cup \langle \Delta_X\rangle\big). \tag{2.12}$$

By the projection formula again

$$P_*\big(\langle \sigma \times X\rangle \cup \langle \Delta_X\rangle\big) = \langle \sigma\rangle.$$

We complete the proof.

□

2.3 Cone operator

In this subsection we construct $\mathbf{Con}^h$, then prove parts (3), (4) in Proposition 1.5. The construction is the key in proving the surjectivity of L^h_{loc}. Let ξ be a C^∞ real valued function on Υ that has the affine coordinate $z \in \mathbb{C}$ as in the introduction, satisfying (a) ξ is a real valued between 0 and 1; (b) $\xi(z) = 1$ for $|z| \leq 1$; (c) $\xi(z) = 0$ in a small neighborhood of the unstable point ∞. We denote its lift to $\Upsilon \times V^h \times X$ also by ξ. For any natural number N, we define a diffeomorphism F_N,

$$\begin{array}{ccc} \Upsilon \times X \times X & \to & \Upsilon \times X \times X \\ (z, \mathbf{x}, \mathbf{y}) & \to & (\frac{z}{N}, \mathbf{x}, \mathbf{y}). \end{array}$$

Let $\pi : \Upsilon \times X \times X \to X(3rd)$ the projection. Let $I_h \subset \Upsilon \times \mathbf{P}^{n+1} \times \mathbf{P}^{n+1-h}$ be the subvariety that is the fibre of the restriction of Ω to $t = 0$ $\Bigg($In coordinates, I_h is the graph of the rational map

$$\left.\begin{array}{ccc} \Upsilon \times \mathbf{P}^{n+1} & \dashrightarrow & \mathbf{P}^{n+1-h} \\ (z, [x_0, \cdots, x_{n+1}]) & \dashrightarrow & [zx_{n+2-h} + x_0, x_1, \cdots, x_{n+h-1}] \end{array}\right).$$

Definition 2.3. *Let $\sigma \in \mathscr{L}_C(V^h)$. We define the cone operator, denoted by $\mathbf{Con}^h(\sigma)$, to be the functional on the test forms ϕ of X,*

$$\phi \quad\longrightarrow\quad \lim_{N\to\infty} \int_{[(\Upsilon\times\sigma\times X)\wedge_{\tau_0} I_h]} F_N^*(\xi)\pi^*(\phi). \tag{2.13}$$

Equivalently, if it exists, $\mathbf{Con}_1^1(\sigma)$ *is defined to the weak limit*

$$\lim_{N\to\infty} \pi_*\left(F_N^*(\xi)[(\Upsilon\times\sigma\times X)\wedge_{\tau_0} I_h]\right)$$

where N *is a natural number.*

Proposition 2.4. *Let* X *be equipped with a de Rham data. Let* $\sigma\subset V^h$ *be a* C^∞ *singular cycle in a general position. Then*

(1) as a functional, $\mathbf{Con}^h(\sigma)$ *is convergent and continuous;*

(2) as a current, $\mathbf{Con}^h(\sigma)$ *is closed;*

(3) as a current, $\mathbf{Con}^h(\sigma)$ *is Lebesgue and satisfies*

$$[\mathbf{Con}^h(\sigma)\wedge V^h] = \sigma. \tag{2.14}$$

(4) If there is a coniveau subgroup of level k *containing the cohomology of* σ*, then there is a coniveau subgroup of the same level* k *containing the cohomology of* $\mathbf{Con}^h(\sigma)$*.*

Proof. We'll make a local computation on the manifold $\Upsilon\times V^h\times X$. So we set up the local coordinates first. Let W be a neighborhood centered around a point $\mathbf{a}_\infty$ in the support of

$$(\Upsilon\times\sigma\times X)\wedge_{\tau_0} I_h.$$

It is sufficient to consider such a point $\mathbf{a}_\infty$ whose projection to Υ is the unstable point ∞. Next we set-up the coordinates for the local expression. Let W be equipped with a standard, C^∞ real coordinates system $\mathbb{R}^{2(2n-h+1)}$. In this chart, we may assume the orthogonal projection of W to V^h contains $|\sigma|$. Let η_i, finite i, be a partition of unity for $\Upsilon\times V^h\times X$ such that η_1 has a compact support in W.

(1) With a suitable de Rham data, we apply the conditional associativity in II, [4] to obtain that

$$\Big[\big[(\Upsilon\times\sigma\times X)\wedge_1(\Upsilon\times V^h\times X)\big]\wedge_{\tau_0} I_h\Big] = \Big[(\Upsilon\times\sigma\times X)\wedge_1\big[(\Upsilon\times V^h\times X)\wedge_{\tau_0} I_h\big]\Big]$$

where the subscript 1 is the intersection in $\Upsilon\times V^h\times X$, and each intersection uses associative de Rham data. Let $\mathcal{I} = \big[(\Upsilon\times V^h\times X)\wedge_{\tau_0} I_h\big]$ which is the integration over the algebraic cycle

$$(\Upsilon\times V^h\times X)\bullet_{\tau_0} I_h$$

denoted also by $\mathcal{I}$. Then the integral in (2.13) can be written as

$$\int_{[(\Upsilon\times\sigma\times X)\wedge\mathcal{I}]} F_N^*(\xi)\pi^*(\phi). \tag{2.15}$$

Using the partition of unity, it suffices to consider the value

$$\int_{\eta_1[(\Upsilon\times\sigma\times X)\wedge\mathcal{I}]} F_N^*(\xi)\pi^*(\phi). \tag{2.16}$$

Claim 2.5. *It suffices to show*

$$\left|\int_{\eta_1[(\Upsilon\times\sigma\times X)\wedge\mathcal{I}]} (F_{N_1}^*(\xi) - F_{N_2}^*(\xi))\pi^*(\phi)\right| \tag{2.17}$$

is sufficiently small when N_1, N_2 are sufficiently large.

Proof. of Claim 2.5: Let $g_N = F_{N_1}^*(\xi) - F_{N_2}^*(\xi)$ for sufficiently large N_1, N_2 where $N = max(N_1, N_2)$. Also we may assume $\pi^*(\phi) = \phi_0 d\mu$ where $d\mu$ is the volume form of the Euclidean coordinates' plane $\mathcal{V}$ in the chart and ϕ_0 is a C^∞ function with the compact support in the chart. Then

$$(2.17) = |\int_{\phi_0 g_N \eta_1[(\Upsilon\times\sigma\times X)\wedge\mathcal{I}]} d\mu|. \tag{2.18}$$

Since the current in the domain of the integral expression (2.18) is Lebesgue, there is a bounded (for all N), integrable function $L_N(y)$ on $\mathcal{V}$ such that

$$(2.18) = |\int_{y\in\mathcal{V}} L_N(y)d\mu|. \tag{2.19}$$

Since the support of g_N approaches the measurable set

$$P_{\mathcal{V}}(\{\infty\}\times\sigma\times X)\cap\tau^{-1}(I_h)$$

where $P_{\mathcal{V}}$ is the localized orthogonal projection from $\Upsilon\times V^h\times X$ to $\mathcal{V}$. Since in a neighborhood of a.e point, it is a semi-algebraic set of a lower real dimension, the set has Lebesgue measure zero. Thus the measure of the support of $L_N(y)$ weakly approaches 0 as $N\to\infty$. Therefore the number (2.18) approaches 0 as $N\to\infty$. This shows the functional is convergent. □

Let $\phi_l, l = 1, 2, \cdots$ be a sequence of test forms, and a_l, a sequence of real numbers approaching ∞ such that $a_l\phi_l$ is bounded for all l. Then by the above argument and boundeness of the Lebesgue currents, the evaluation of the functional

$$\int_{\mathbf{Con}^h(\sigma)} a_l\phi_l \tag{2.20}$$

is also bounded. This implies $\int_{\mathbf{Con}^h(\sigma)}\phi_l \to 0$ as $l\to\infty$. Therefore the functional

$$\phi \quad\longrightarrow\quad \lim_{N\to\infty}\int_{[(\Upsilon\times\sigma\times X)\wedge_{\tau_0} I_h]} F_N^*(\xi)\pi^*(\phi),$$

denoted by $\mathbf{Con}^h(\sigma)$, is continuous, therefore a current.

(2) To show the current $\mathbf{Con}^h(\sigma)$ is closed, it suffices to show the functional

$$\lim_{N\to\infty}\int_{[(\Upsilon\times\sigma\times X)\wedge_{\tau_0} I_h]} F_N^*(\xi)\pi^*(d\phi) \tag{2.21}$$

is zero, where ϕ is a C^∞ form on X. Notice

$$\begin{aligned} &\lim_{N\to\infty}\int_{[(\Upsilon\times\sigma\times X)\wedge_{\tau_0} I_h]} F_N^*(\xi)\pi^*(d\phi) \\ &= \lim_{N\to\infty}\int_{d[(\Upsilon\times\sigma\times X)\wedge_{\tau_0} I_h]} F_N^*(\xi)\pi^*(\phi) \\ &-\lim_{N\to\infty}\int_{\left[F_N^*(d\xi)\wedge[(\Upsilon\times\sigma\times X)\wedge_{\tau_0} I_h]\right]} \pi^*(\phi). \end{aligned} \tag{2.22}$$

Then we observe the current

$$\left[F_N^*(d\xi)\wedge[(\Upsilon\times\sigma\times X)\wedge_{\tau_0} I_h]\right]$$

to see it is the same type of currents as in the domain of (2.18). Precisely, its projection to the plane $\mathcal{V}$ is the measure approaching 0 as $N\to\infty$. So

$$\lim_{N\to\infty}\int_{\left[F_N^*(d\xi)\wedge[(\Upsilon\times\sigma\times X)\wedge_{\tau_0} I_h]\right]} \pi^*(\phi) = 0. \tag{2.23}$$

For the integral

$$\int_{d[(\Upsilon\times\sigma\times X)\wedge_{\tau_0} I_h]} F_N^*\Big(\xi\pi^*\phi\Big)$$

we apply the Leibniz rule II, [4] to obtain the current $d[(\Upsilon\times\sigma\times X)\wedge_{\tau_0} I_h]$ is also zero. Hence (2.21) is zero.

(3) In above arguments, we obtain a closed current $\mathbf{Con}^h(\sigma)$. To show it is Lebesgue, it suffices to show it satisfies Randon-Nilkodym condition. We continue the setting as in part (1) to let $\lambda\to 0$ be positive real numbers. The Radon-Nilkodym condition on $\mathbf{Con}^h(\sigma)$ is the convergence of

$$\int_{\eta_1[(\Upsilon\times\sigma\times X)\wedge\mathcal{I}]} F_N^*(\xi)\pi^*(S_\lambda^*(\phi)), \tag{2.24}$$

as $\lambda\to 0$, where S_λ is the local operator in the chart, sending

$$y\to y\lambda^{-1}.$$

for the variable y of the coordinate's plane $\mathcal{V}$. It is equivalent to the assertion that

$$\int_{\eta_1[(\Upsilon\times\sigma\times X)\wedge\mathcal{I}]} F_N^*(\xi)\pi^*\big(S_{\lambda_1}^*(\phi)-S_{\lambda_2}^*(\phi)\big), \tag{2.25}$$

converges to 0 as $\lambda_i \to 0$ for $i = 1, 2$. Since

$$[(\Upsilon \times \sigma \times X) \wedge \mathcal{I}]$$

is Lebesgue, therefore satisfies the Radon-Nilkodym condition and $\eta_1, F_N^*(\xi)$ are both bounded (also for all N), (2.25) indeed converges to 0. Therefore $\mathbf{Con}^h(\sigma)$ is a Lebesgue current. Next we consider its intersection. Let V^h be the plane section of codimension h, ϕ a test form on X, We calculate

$$\begin{aligned}
&\int_{[\mathbf{Con}^h(\sigma) \wedge V^h]} \phi \\
&= \lim_{\lambda \to 0} \int_{\mathbf{Con}^h(\sigma)} R_\lambda^X(V^h) \wedge \phi \\
&= \lim_{\lambda \to 0} \lim_{N \to \infty} \int_{[(\Upsilon \times \sigma \times X) \wedge_{\tau_0} I_h]} (F_N)^*(\xi) \pi^* \Big(R_\lambda^X(V^h) \wedge \phi \Big)
\end{aligned} \tag{2.26}$$

where $\pi^* \Big(R_\lambda^X(V^h) \Big) = R_\lambda^{(\Upsilon \times V^h \times X)}(\Upsilon \times V^h \times V^h)$ for suitable de Rham data.

Next we show the key step that the iterated limit in (2.26) is independent of the limit's order, i.e the order is interchangeable. We'll make a local computation which continue with the setting in (1). The localized integral in (2.26) in the neighborhood of $supp(\eta_1)$ is equal to

$$\int_{F_N^*(\xi)\eta_1[(\Upsilon \times \sigma \times X) \wedge_{\tau_0} I_h] \wedge \pi^*(\phi)} \pi^* \Big(R_\lambda^X(V^h) \Big) \tag{2.27}$$

Let λ_1, λ_2 be two numbers near 0. By the same reason for the convergence of the Radon-Nilkodym condition of $\mathbf{Con}^h(\sigma)$, in particular the boundeness of $F_N^*(\xi)$,

$$\int_{F_N^*(\xi)\eta_1[(\Upsilon \times \sigma \times X) \wedge_{\tau_0} I_h] \wedge \pi^*(\phi)} \pi^* \Big(R_{\lambda_1}^X(V^h) - R_{\lambda_1}^X(V^h) \Big)$$

converges to 0 uniformly for all N. Hence the convergence of (2.27) is uniform for all N. Then Lemma B.1 implies the iterated limit in (2.26) is independent of the limit's order. So we obtain

$$\begin{aligned}
(2.26) &= \lim_{N \to \infty} \lim_{\lambda \to 0} \int_{[(\Upsilon \times \sigma \times X) \wedge_{\tau_0} I_h]} F_N^*(\xi) \pi^* \Big(R_\lambda^X(V^h) \wedge \phi \Big) \\
&= \lim_{N \to \infty} \lim_{\lambda \to 0} \int_{[(\Upsilon \times \sigma \times X) \wedge_{\tau_0} I_h]} \pi^*(R_\lambda^X(V^h)) \wedge F_N^*(\xi) \pi^*(\phi) \\
&= \lim_{N \to \infty} \int_{\Big[[(\Upsilon \times \sigma \times X) \wedge_{\tau_0} I_h] \wedge (\Upsilon \times V^h \times V^h) \Big]} F_N^*(\xi) \pi^*(\phi) \\
&= \lim_{N \to \infty} \int_{F_N^*(\xi) \Big[[(\Upsilon \times \sigma \times X) \wedge \mathcal{I} \Big]} \pi^*(\phi)
\end{aligned} \tag{2.28}$$

Then we can continue

$$
\begin{aligned}
(2.28) &= \lim_{N\to\infty} \int_{F_N^*(\xi)\left[(\Upsilon\times\sigma\times X)\wedge\mathcal{I}\right]} \pi^*(\phi) \\
&= \int_{\left[\left((\Upsilon\times\sigma\times X)\wedge\mathcal{I}\right)\right]} \pi^*\phi \qquad (2.29)\\
&- \lim_{N\to\infty} \int_{\left[\left(1-F_N^*(\xi)\right)[\Upsilon\wedge(\Upsilon\times\sigma\times X)]\wedge\mathcal{I}\right)\right]} \pi^*\phi.
\end{aligned}
$$

Now we consider

$$
\lim_{N\to\infty} \int_{\left[\left(1-F_N^*(\xi)\right)[\Upsilon\wedge(\Upsilon\times\sigma\times X)]\wedge\mathcal{I}\right)\right]} \pi^*\phi. \tag{2.30}
$$

Since $1-\xi(\frac{z}{N})$ is a function supported on $\{z : |z| \geq N\}$, the Lebesgue measure of the orthogonal projection of

$$
(1-F_N^*(\xi))\Big[[\Upsilon\wedge(\Upsilon\times\sigma\times X)]\wedge\mathcal{I}\Big]
$$

to the plane $\mathcal{V}$ approaches to 0, as $N\to 0$. By the same reason for Claim 2.5, i.e. measure's weak convergence, (2.30) is 0. Hence

$$
(2.28) = \int_{\left[(\Upsilon\times\sigma\times X)\wedge\mathcal{I}\right]} \pi^*\phi \tag{2.31}
$$

Let

$$
P_h : \Upsilon\times V^h\times X \to V^h\times X
$$

be the projection. We observe

$$
(P_h)_*(\mathcal{I}) = \Delta_{V^h},
$$

where Δ_{V^h} is the diagonal of V^h. By the projection formula for the projection P_h, we obtain that

$$
(2.31) = \int_{[(\sigma\times V^h)\wedge\Delta_{V^h}]} (P_X)^*(\phi) \tag{2.32}
$$

where $P_X : X\times X \to X(2rd)$ is the projection map. By the commutativity of the intersection,

$$
\begin{aligned}
&\int_{[(\sigma\times V^h)\wedge\Delta_{V^h}]} (P_X)^*(\phi) \\
&= \int_{\left[\Delta_{V^h}\wedge[(\sigma\times V^h)\right]} (P_X)^*(\phi) \qquad (2.33)\\
&= \int_\sigma \phi.
\end{aligned}
$$

We obtain

$$[\mathbf{Con}^h(\sigma) \wedge V^h] = \sigma. \tag{2.34}$$

(4) It suffices to prove it for the smallest algebraic set containing $|\sigma|$. Assume the codimension of $|\sigma| \subset \overline{|\sigma|}$ is k. Let A be the algebraic set

$$\nu\bigg((\Upsilon \times \overline{|\sigma|} \times X) \cap I_h\bigg) \tag{2.35}$$

where $\nu : \Upsilon \times V^h \times X \to X$ is the projection. If the intersection in (2.35) is proper and the projection ν is restricted to generically finite-to-one map, $dim(A)$ is $dim(\overline{|\sigma|}) + 2h$. So $\mathbf{Con}^h(\sigma)$ is contained in A with codimension k, i.e.

$$2dim(A) - dim(\mathbf{Con}^h(\sigma)) = k.$$

The assertion is proved. If either the intersection in (2.35) is not proper or ν is not generically finite to one, then $\mathbf{Con}^h(\sigma)$ is 0 whose cohomology lies in any subgroups.

□

With the constructed $\mathbf{Con}^h$, Proposition 2.4 implies parts (3), (4) in Proposition 1.5. So we complete the proof of Proposition 1.5.

Appendix A Current's version of the coniveau filtration

Leveled filtration (0.1) in the abstract is the re-grouped coniveau filtration. While we review the coniveau filtration below, we'll give another description by using currents. The notion is originated from [2] where Grothendieck proposed a filtration $Filt'^p$, and wrote "Arithemetic filtration ($Filt'^p$) embodies deep arithmetic properties of the scheme ". This later was referred to as the coniveau filtration. The subgroup in the filtration is defined as the linear span of kernels of the linear maps

$$H^{2p+k}(X;\mathbb{Q}) \quad \to \quad H^{2p+k}(X - W;\mathbb{Q}) \tag{A.1}$$

for subvarieties W of codimension at least p. We'll use another interpretation of the coniveau filtration. It is through currents, which are known to unite both homology and cohomology. The definition geometrically uses the notion of support. Let $\mathcal{D}'(X)$ be the space of currents on X. Let $C\mathcal{D}'(X)$ be its subset of closed currents and $E\mathcal{D}'(X)$ be its subset of exact currents. Then de Rham theory is the equality

$$\frac{C\mathcal{D}'(X)}{E\mathcal{D}'(X)} = H(X;\mathbb{C}). \tag{A.2}$$

Next we define another type subgroups. Let $\mathcal{C}^pH^{2p+k}(X)$ be the subgroup of $H^{2p+k}(X)$, whose elements are represented by closed currents supported in some subvariety of codimension at least p. We have the following description.

Lemma A.1. *Let X be a smooth projective variety over $\mathbb{C}$. Then*

$$\mathcal{C}^pH^{2p+k}(X) = N^pH^{2p+k}(X). \tag{A.3}$$

Proof. We recall the coniveau subgroup $N^pH^{2p+k}(X)$ is defined as a subgroup

$$\bigcup_{cod(W)\geq p} ker\Big\{H^{2p+k}(X;\mathbb{Q}) \to H^{2p+k}(X\backslash W;\mathbb{Q})\Big\}.$$

Thus it suffices to focus on the interpretation of $H^{2p+k}(X;\mathbb{Q}) \to H^{2p+k}(X\backslash W;\mathbb{Q})$. It is clear our definition of it is geometric. Hence by the de Rham theory

$$\mathcal{C}^pH^{2p+k}(X) \subset N^pH^{2p+k}(X). \tag{A.4}$$

For the converse, let $\alpha \in N^pH^{2p+k}(X)$. Applying Cor. 8.2.8, [1], we obtain a representative in a singular cycle σ such that $supp(\sigma)$ lies on a subvariety W of $cod(W) \geq p$. Since σ is a current, $\alpha \in \mathcal{C}^pH^{2p+k}(X)$.

□

Appendix B Iterated limits

Lemma B.1. *Let $a_{\lambda_1\lambda_2}$ be the real numbers indexed by two real numbers $\lambda_1 > 0, \lambda_2 > 0$. Assume*

(1) limits $\lim\limits_{\lambda_1\to 0} a_{\lambda_1\lambda_2}$ and $\lim\limits_{\lambda_2\to 0} a_{\lambda_1\lambda_2}$ exist,

(2) the convergence of $\lim\limits_{\lambda_2\to 0} a_{\lambda_1\lambda_2}$ is uniform,

(3) the iterated limit $\lim\limits_{\lambda_2\to 0}\lim\limits_{\lambda_1\to 0} a_{\lambda_1\lambda_2}$ exists.

Then the iterated limit in the other order $\lim\limits_{\lambda_1\to 0}\lim\limits_{\lambda_2\to 0} a_{\lambda_1\lambda_2}$ also exists and

$$\lim_{\lambda_1\to 0}\lim_{\lambda_2\to 0} a_{\lambda_1\lambda_2} = \lim_{\lambda_2\to 0}\lim_{\lambda_1\to 0} a_{\lambda_1\lambda_2}. \tag{B.1}$$

Proof. Let $L = \lim\limits_{\lambda_2\to 0}\lim\limits_{\lambda_1\to 0} a_{\lambda_1\lambda_2}$. Let $\epsilon > 0$ be a real number. By the convergence in the assumptions,

$$|\lim_{\lambda_1\to 0} a_{\lambda_1\lambda_2} - L| \leq \epsilon \tag{B.2}$$

for a small fixed λ_2;

$$|a_{\lambda_1\lambda_2} - \lim_{\lambda_1\to 0} a_{\lambda_1\lambda_2}| \leq \epsilon \tag{B.3}$$

for all sufficiently small λ_1 (note λ_2 is a fixed positive number);

$$|\lim_{\lambda_2\to 0} a_{\lambda_1\lambda_2} - a_{\lambda_1\lambda_2}| \leq \epsilon \tag{B.4}$$

for all sufficiently small λ_1 and another fixed λ_2 (due to the uniform convergence). Combining $B.2$, $B.3$, $B.4$, we obtain for all sufficiently small λ_1,

$$|\lim_{\lambda_2\to 0} a_{\lambda_1\lambda_2} - L| \leq 3\epsilon. \tag{B.5}$$

This completes the proof.

□

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